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Intuitionistic fuzzifications of ideals in BG-algebras

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Abstract

The aim of this paper is to apply the concept of an intuitionistic fuzzy set to ideals and closed ideals in BG-algebras. The notion of an intuitionistic fuzzy closed ideal of a BG-algebra is introduced, and some related properties are investigated. Also, the product of intuitionistic fuzzy BG-algebra is investigated.

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1 Introduction

Zadeh introduced initiatively the concept of fuzzy sets [18] in 1965, from then on some researches have been conducted on the generalization of the notion of fuzzy sets. Atanassov presented a generalization of the notion of fuzzy sets: the concept of intuitionistic fuzzy sets [2] in 1986, and some basic and main results on intuitionistic fuzzy sets were discussed in [2-4].

In 1966, Imai and Iseki [7] introduced two classes of abstract algebra: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. In [6] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebra is a proper subclass of the BCH-algebras. Neggers and Kim [11] introduced a new notion, called a B-algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras. Cho and Kim [5] discussed further relations between B-algebras and other topics especially quasigroups. Kim and Kim [9] introduced the notion of BG-algebras, which is a generalization of B-algebras. Ahn and Lee [1] fuzzified BG-algebras. Saeid [13] introduced fuzzy topological BG-algebras. In the same year Zarandi and Saeid [20] presented intuitionistic fuzzy ideals of BG-algebras. Senapati et al. [15] presented the concept and basic properties of intuitionistic fuzzy BG- subalgebras.

In this paper, some extended result of intuitionistic fuzzy ideal called IFC-ideal of BG-algebras is presented based on intuitionistic fuzzy sets(IFSs), and obtain some results on them. At the same time, the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BG-algebra are presented and investigated some related properties. The product of intuitionistic fuzzy BG-algebra has been introduced and some important properties are of it are also studied.

The rest of this paper is organized as follows. The following section briefly reviews some background on BG-algebra, BG-subalgebra, intuitionistic fuzzy set, intuitionistic fuzzy BG-subalgebras. In Section 3, we propose the concepts and operations of intuitionistic fuzzy ideal and intuitionistic fuzzy closed ideal (IFC-ideal) and discuss their properties in detail. In Section 4, we investigate properties of intuitionistic fuzzy ideals under homomorphisms. In Section 5, we introduce equivalence relations on intuitionistic fuzzy ideals.In Section 6, product of intuitionistic fuzzy BG-algebra and some of its properties are studied. Finally, in Section 7, we draw the conclusion and present some topics for future research.

2 Preliminaries

This section contains some definitions and results to be used in the sequel. The BG-algebra is a very important branch of a modern algebra, which is defined by Kim and Kim [9]. This algebra is defined as follows.

Definition 2.1 [9]*(BG-algebra) A non-empty set X with a constant 0 and a binary operation $*$ is said to be BG-Algebra if it satisfies the following axioms*

1. $x * x = 0$
2. $x * 0 = x$
3. $(x * y) * (0 * y) = x$, *for all* $x, y \in X$.

A BG-algebra is denoted by $(X, *, 0)$. An example of BG-algebra is given below.

Example 2.2 *Let* $X = \{0, 1, 2, 3, 4, 5\}$ *be a set. The binary operation* $*$ *over* X *is defined as*

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	5	3	4
2	2	1	0	4	5	3
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

This table satisfies all the conditions of Definition 2.1. Hence, $(X, *, 0)$ is a BG-algebra.

Now, we introduce the concept of BG-subalgebra over a crisp set X and the binary operation $*$ in the following. The definition of BG-subalgebra is given below.

Definition 2.3 [9]*(BG-subalgebra) A non-empty subset S of a BG-algebra X is called a subalgebra of X if* $x * y \in S$, *for all* $x, y \in S$.

From this definition it is observed that, if a subset S of a BG-algebra satisfies only the closer property, then S becomes a BG-subalgebra.

Definition 2.4 [18]*(Fuzzy Set) Let X be the collection of objects denoted generally by x then a fuzzy set A in X is defined as* $A = \{< x, \alpha_A(x) >: x \in X\}$ *where $\alpha_A(x)$ is called the membership value of x in A and* $0 \leq \alpha_A(x) \leq 1$.

Combined the definition of BG-subalgebra over crisp set and the idea of fuzzy set Ahn and Lee [1] defined fuzzy BG-subalgebra, which is defined below.

Definition 2.5 [1]*(Fuzzy BG-subalgebra) Let* $A = \{< x, \alpha_A(x) >: x \in X\}$ *be a fuzzy set in a BG-algebra. Then A is called a fuzzy subalgebra of X if* $\alpha_A(x*y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$ *for all* $x, y \in X$, *where $\alpha_A(x)$ is the membership value of x in X.*

Definition 2.6 [10]*(Fuzzy BG-ideal) A fuzzy set* $A = \{< x, \alpha_A(x) >: x \in X\}$ *in* X *is called a fuzzy ideal of* X *if it satisfies (i)* $\alpha_A(0) \geq \alpha_A(x)$ *and (ii)* $\alpha_A(x) \geq min\{\alpha_A(x * y), \alpha_A(y)\}$ *for all* $x, y \in X$.

In a fuzzy set only the membership value $\alpha_A(x)$ of an element x is considered, and the non-membership value can be taken as $1 - \alpha_A(x)$. This value also lies between 0 and 1. But in reality this is not true for all cases, i.e, the non-membership value may be strictly less than 1. This idea was first incorporated by Atanassov [2] and initiated the concept of intuitionistic fuzzy set defined below.

Definition 2.7 [2]*(IFS) An intuitionistic fuzzy set* A *over* X *is an object having the form* $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\}$*, where* $\alpha_A(x) : X \to [0, 1]$ *and* $\beta_A(x) : X \to [0, 1]$*, with the condition* $0 \leq \beta_A(x)+\beta_A(x) \leq 1$ *for all* $x \in X$*. The numbers* $\alpha_A(x)$ *and* $\beta_A(x)$ *denote, respectively, the degree of membership and the degree of non-membership of the element* x *in the set* A*. For the sake of simplicity, we shall use the symbol* $A = (\alpha_A, \beta_A)$ *for the intuitionistic fuzzy subset* $A = \{\langle x, \alpha_A(x), \beta_A(x)\rangle : x \in X\}$.

Extending the idea of fuzzy BG-subalgebra, Zarandi and Saeid [20] defined intuitionistic fuzzy BG-subalgebra. In intuitionistic fuzzy BG-subalgebra, two conditions are to be satisfied, instead of one condition in fuzzy BG-subalgebra.

Definition 2.8 [15]*(Intuitionistic fuzzy BG-subalgebra) An IFS* $A = (\alpha_A, \beta_A)$ *in* X *is called an intuitionistic fuzzy subalgebra of* X *if it satisfies the following two conditions,*

(IBS1) $\alpha_A(x * y) \geq \min\{\alpha_A(x), \alpha_A(y)\}$

and (IBS2) $\beta_A(x * y) \leq \max\{\beta_A(x), \beta_A(y)\}$.

3 IFC-ideals of BG-algebras

In this section, intuitionistic fuzzy ideal and IFC-ideal of BG-algebra are defined and some propositions and theorems are presented. In what follows, let X denote a BG-algebra unless otherwise specified.

Definition 3.1 [20] An IFS $A = (\alpha_A, \beta_A)$ in X is called an intuitionistic fuzzy ideal of X if it satisfies:

(IBS3) $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$

(IBS4) $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$

(IBS5) $\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\}$

for all $x, y \in X$.

Example 3.2 *Consider a BG-algebra* $X=\{0,1,2,3\}$ *with the following Cayley table*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let $A = (\alpha_A, \beta_A)$ *be an IFS in* X *defined as* $\alpha_A(0) = \alpha_A(2) = 1$, $\alpha_A(1) = \alpha_A(3) = m$, $\beta_A(0) = \beta_A(2) = 0$ *and* $\beta_A(1) = \beta_A(3) = n$, *where* $m, n \in [0, 1]$ *and* $m + n \leq 1$. *Then* $A = (\alpha_A, \beta_A)$ *is an intuitionistic fuzzy ideal of* X.

Definition 3.3 *An IFS* $A = (\alpha_A, \beta_A)$ *in* X *is called an IFC-ideal of* X *if it satisfies (IBS4), (IBS5) along with (IBS6)* $\alpha_A(0 * x) \geq \alpha_A(x)$ *and* $\beta_A(0 * x) \leq \beta_A(x)$, *for all* $x \in X$.

Example 3.4 *Consider a BG-algebra* $X=\{0,1,2,3,4,5\}$ *with the table in Example 2.2. We define an IFS* $A = (\alpha_A, \beta_A)$ *in* X *by,* $\alpha_A(0) = 0.6$, $\alpha_A(1) = \alpha_A(2) = 0.5$, $\alpha_A(3) = \alpha_A(4) = \alpha_A(5) = 0.35$, $\beta_A(0) = 0.15$, $\beta_A(1) = \beta_A(2) = 0.3$ *and* $\beta_A(3) = \beta_A(4) = \beta_A(5) = 0.5$. *By routine calculations, one can verify that* $A = (\alpha_A, \beta_A)$ *is an IFC-ideal of* X.

Proposition 3.5 *Every IFC-ideal is an intuitionistic fuzzy-ideal.*

The converse of above Proposition is not true in general as seen in the following example.

Example 3.6 *Consider a BG-algebra* $X = \{0, 1, 2, 3, 4, 5\}$ *with the following table*

$*$	0	1	2	3	4	5
0	0	5	4	3	2	1
1	1	0	5	4	3	2
2	2	1	0	5	4	3
3	3	2	1	0	5	4
4	4	3	2	1	0	5
5	5	4	3	2	1	0

Let us define an IFS $A = (\alpha_A, \beta_A)$ *in* X *by* $\alpha_A(0) = 0.6$, $\alpha_A(1) = 0.5$, $\alpha_A(2) = \alpha_A(3) = \alpha_A(4) = \alpha_A(5) = 0.35$, $\beta_A(0) = 0.15$, $\beta_A(1) = 0.3$, *and* $\beta_A(2) = \beta_A(3) = \beta_A(4) = \beta_A(5) = 0.5$. *We know that* $A = (\alpha_A, \beta_A)$ *is an intuitionistic fuzzy ideal of* X. *But it is not an IFC-ideal of* X *since* $\alpha_A(0 * x) \ngeq \alpha_A(x)$ *and* $\beta_A(0 * x) \nleq \beta_A(x)$.

Corollary 3.7 *Every intuitionistic fuzzy BG-subalgebra satisfying (IBS4) and (IBS5) is an IFC-ideal.*

Theorem 3.8 *Every IFC-ideal of a BG-algebra X is an intuitionistic fuzzy BG-subalgebra of X.*

Proof: If $A = (\alpha_A, \beta_A)$ is an IFC-ideal of X, then for any $x \in X$ we have $\alpha_A(0 * x) \geq \alpha_A(x)$ and $\beta_A(0 * x) \leq \beta_A(x)$. Now

$$\begin{aligned} \alpha_A(x*y) &\geq \min\{\alpha_A((x*y)*(0*y)), \alpha_A(0*y)\}, \text{ by (IBS4)} \\ &= \min\{\alpha_A(x), \alpha_A(0*y)\} \\ &\geq \min\{\alpha_A(x), \alpha_A(y)\}, \text{ by (IBS6)} \\ \text{and} \quad \beta_A(x*y) &\leq \max\{\beta_A((x*y)*(0*y)), \beta_A(0*y)\}, \text{ by (IBS5)} \\ &= \max\{\beta_A(x), \beta_A(0*y)\} \\ &\leq \max\{\beta_A(x), \beta_A(y)\}, \text{ by (IBS6).} \end{aligned}$$

Hence the theorem. □

Proposition 3.9 *If an IFS $A = (\alpha_A, \beta_A)$ in X is an IFC-ideal, then for all $x \in X$, $\alpha_A(0) \geq \alpha_A(x)$ and $\beta_A(0) \leq \beta_A(x)$.*

Proof: Straightforward. □

Definition 3.10 [2] Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two IFSs on X. Then the intersection of A and B is denoted by $A \cap B$ and is given by $A \cap B = \{\min(\alpha_A, \alpha_B), \max(\beta_A, \beta_B)\}$. Also the complement of A is denoted by $\overline{A}$ and is defined by $\overline{A} = (\beta_A, \alpha_A)$.

The intersection of two intuitionistic fuzzy ideals of a BG-algebras is also an intuitionistic fuzzy ideal of a BG-algebra, which is proved in the following theorem.

Theorem 3.11 *Let A_1 and A_2 be two intuitionistic fuzzy ideals of a BG-algebras X. Then $A_1 \cap A_2$ is also intuitionistic fuzzy ideal of BG-algebra X.*

Proof: Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2. Now, $\alpha_{A_1 \cap A_2}(0) = \alpha_{A_1 \cap A_2}(x * x) \geq \min\{\alpha_{A_1 \cap A_2}(x), \alpha_{A_1 \cap A_2}(x)\} = \alpha_{A_1 \cap A_2}(x)$ and $\beta_{A_1 \cap A_2}(0) = \beta_{A_1 \cap A_2}(x * x) \leq \max\{\beta_{A_1 \cap A_2}(x), \beta_{A_1 \cap A_2}(x)\} = \beta_{A_1 \cap A_2}(x)$. Also,

$$\begin{aligned} \alpha_{A_1\cap A_2}(x) &= \min\{\alpha_{A_1}(x), \alpha_{A_2}(x)\} \\ &\geq \min\{\min\{\alpha_{A_1}(x*y), \alpha_{A_1}(y)\}, \min\{\alpha_{A_2}(x*y), \alpha_{A_2}(y)\}\} \\ &= \min\{\min\{\alpha_{A_1}(x*y), \alpha_{A_2}(x*y)\}, \min\{\alpha_{A_1}(y), \alpha_{A_2}(y)\}\} \\ &= min\{\alpha_{A_1\cap A_2}(x*y), \alpha_{A_1\cap A_2}(y)\} \\ \text{and } \beta_{A_1\cap A_2}(x) &= \max\{\beta_{A_1}(x), \beta_{A_2}(x)\}, \\ &\leq \max\{\max\{\beta_{A_1}(x*y), \beta_{A_1}(y)\}, \max\{\beta_{A_2}(x*y), \beta_{A_2}(y)\}\} \\ &= \max\{\max\{\beta_{A_1}(x*y), \beta_{A_2}(x*y)\}, \max\{\beta_{A_1}(y), \beta_{A_2}(y)\}\} \\ &= max\{\beta_{A_1\cap A_2}(x*y), \beta_{A_1\cap A_2}(y)\} \end{aligned}$$

Hence, $A_1 \cap A_2$ is an intuitionistic fuzzy ideal of a BG-algebra of X. □

The above theorem can be generalized as follows.

Theorem 3.12 *Let $\{A_i | i = 1, 2, 3, 4, \ldots\}$ be a family of intuitionistic fuzzy ideals of a BG-algebra X. Then $\bigcap A_i$ is also an intuitionistic fuzzy ideal of BG-algebra X where, $\bigcap A_i = (\min \alpha_{A_i}(x), \max \beta_{A_i}(x))$.*

In the same way and by the definition of $\overline{A}$ we can prove the following result.

Theorem 3.13 *An IFS $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X if and only if the fuzzy sets α_A and $\overline{\beta}_A$ are fuzzy ideals of X.*

Proof: Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of X. Clearly, α_A is a fuzzy ideal of X. For every $x, y \in X$, we have $\overline{\beta}_A(0) = 1 - \beta_A(0) \geq 1 - \beta_A(x) = \overline{\beta}_A(x)$ and $\overline{\beta}_A(x) = 1 - \beta_A(x) \geq 1 - max\{\beta_A(x * y), \beta_A(y)\} = min\{1 - \beta_A(x * y), 1 - \beta_A(y)\} = min\{\overline{\beta}_A(x * y), \overline{\beta}_A(y)\}$. Hence, $\overline{\beta}_A$ is a fuzzy ideal of X.

Conversely, assume that α_A and $\overline{\beta}_A$ are fuzzy ideals of X. For every $x, y \in X$, we get $\alpha_A(0) \geq \alpha_A(x)$ and $\overline{\beta}_A(0) \geq \overline{\beta}_A(x)$. This implies, $1 - \beta_A(0) \geq 1 - \beta_A(x)$. That is, $\beta_A(0) \leq \beta_A(x)$. Also $\alpha_A(x) \geq min\{\alpha_A(x * y), \alpha_A(y)\}$ and $1 - \beta_A(x) = \overline{\beta}_A(x) \geq \min\{\overline{\beta}_A(x * y), \overline{\beta}_A(y)\} = \min\{1 - \beta_A(x * y), 1 - \beta_A(y)\} = 1 - \max\{\beta_A(x * y), \beta_A(y)\}$. That is, $\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\}$. Hence, $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X. □

Lemma 3.14 [20] Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of X. If $x * y \leq z$ then $\alpha_A(x) \geq \min\{\alpha_A(y), \alpha_A(z)\}$ and $\beta_A(x) \leq \max\{\beta_A(y), \beta_A(z)\}$.

Lemma 3.15 [20] Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy ideal of X. If $x \leq y$ then $\alpha_A(x) \geq \alpha_A(y)$ and $\beta_A(x) \leq \beta_A(y)$ i.e., α_A is order-reserving and β_A is order-preserving.

The above lemma can be generalized as

Lemma 3.16 *If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X, then $(\ldots.((x * a_1) * a_2) * \ldots.) * a_n = 0$ for any $x, a_1, a_2, \ldots., a_n \in X$, implies $\alpha_A(x) \geq \min\{\alpha_A(a_1), \alpha_A(a_2), \ldots.., \alpha_A(a_n)\}$; $\beta_A(x) \leq \max\{\beta_A(a_1), \beta_A(a_2), \ldots.., \beta_A(a_n)\}$.*

Proof: Using induction on n and by Lemma 1 and Lemma 2 we can easily prove the theorem. □

We define two operators $\bigoplus A$ and $\bigotimes A$ on IFS as follows:

Definition 3.17 *Let $A = (\alpha_A, \beta_A)$ be an IFS defined on X. The operators $\bigoplus A$ and $\bigotimes A$ are defined as $\bigoplus A = (\alpha_A(x), \overline{\alpha}_A(x))$ and $\bigotimes A = (\overline{\beta}_A(x), \beta_A(x))$ in X.*

Theorem 3.18 *If $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy ideal of X, then (i) $\bigoplus A$, and (ii) $\bigotimes A$, both are intuitionistic fuzzy ideals of X.*

Proof: For (i), it is sufficient to show that $\overline{\alpha}_A$ satisfies the second part of the conditions (IBS3) and (IBS5). We have $\overline{\alpha}_A(0) = 1 - \alpha_A(0) \leq 1 - \alpha_A(x) \leq \overline{\alpha}_A(x)$. Let $x, y \in X$. Then $\overline{\alpha}_A(x) = 1 - \alpha_A(x) \leq 1 - \min\{\alpha_A(x * y), \alpha_A(y)\} = \max\{1 - \alpha_A(x * y), 1 - \alpha_A(y)\} = \max\{\overline{\alpha}_A(x * y), \overline{\alpha}_A(y)\}$. Hence, $\bigoplus A$ is an intuitionistic fuzzy ideal of X.
For (ii), it is sufficient to show that $\overline{\beta}_A$ satisfies the first part of the conditions (IBS3) and (IBS4). We have $\overline{\beta}_A(0) = 1 - \beta_A(0) \geq 1 - \beta_A(x) \geq \overline{\beta}_A(x)$. Let $x, y \in X$. Then $\overline{\beta}_A(x) = 1 - \beta_A(x) \geq 1 - \max\{\beta_A(x * y), \beta_A(y)\} = \min\{1 - \beta_A(x * y), 1 - \beta_A(y)\} = \min\{\overline{\beta}_A(x * y), \overline{\beta}_A(y)\}$. Hence, $\bigotimes A$ is an intuitionistic fuzzy ideal of X. □

Definition 3.19 [15] Let $A = (\alpha_A, \beta_A)$ is an intuitionistic fuzzy BG-subal gebra of X. For $s, t \in [0, 1]$, the set $U(\alpha_A : s) = \{x \in X : \alpha_A(x) \geq s\}$ is called upper s-level of A and $L(\beta_A : t) = \{x \in X : \beta_A(x) \leq t\}$ is called lower t-level of A.

Theorem 3.20 *An IFS $A = (\alpha_A, \beta_A)$ is an IFC-ideal of X if and only if the sets $U(\alpha_A : s)$ and $L(\beta_A : t)$ are closed ideal of X for every $s, t \in [0, 1]$.*

Proof: Suppose that $A = (\alpha_A, \beta_A)$ is an IFC-ideal of X. For $s \in [0, 1]$, obviously, $0 * x \in U(\alpha_A : s)$, where $x \in X$. Let $x, y \in X$ be such that $x*y \in U(\alpha_A : s)$ and $y \in U(\alpha_A : s)$. Then $\alpha_A(x) \geq \min\{\alpha_A(x*y), \alpha_A(y)\} \geq s$. Then $x \in U(\alpha_A : s)$. Hence, $U(\alpha_A : s)$ is closed ideal of X.

For $t \in [0, 1]$, obviously, $0 * x \in L(\beta_A : t)$. Let $x, y \in X$ be such that $x * y \in L(\beta_A : t)$ and $y \in L(\beta_A : t)$. Then $\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\} \leq t$. Then $x \in L(\beta_A : t)$. Hence, $L(\beta_A : t)$ is closed ideal of X.

Conversely, assume that each non-empty level subset $U(\alpha_A : s)$ and $L(\beta_A : t)$ are closed ideals of X. For any $x \in X$, let $\alpha_A(x) = s$ and $\beta_A(x) = t$. Then $x \in U(\alpha_A : s)$ and $x \in L(\beta_A : t)$. Since $0 * x \in U(\alpha_A : s) \cap L(\beta_A : t)$, it follows that $\alpha_A(0 * x) \geq s = \alpha_A(x)$ and $\beta_A(x) \leq t = \beta_A(x)$, for all $x \in X$.

If there exist $\lambda, \kappa \in X$ such that $\alpha_A(\lambda) < \min\{\alpha_A(\lambda * \kappa), \alpha_A(\kappa)\}$, then by taking $s' = \frac{1}{2}\Big[\alpha_A(\lambda*\kappa) + \min\{\alpha_A(\lambda), \alpha_A(\kappa)\}\Big]$, it follows that $\lambda*\kappa \in U(\alpha_A : s')$ and $\kappa \in U(\alpha_A : s')$, but $\lambda \notin U(\alpha_A : s')$, which is a contradiction. Hence, $U(\alpha_A : s')$ is not closed ideal of X.

Again, if there exist $\gamma, \delta \in X$ such that $\beta_A(\gamma) > \max\{\beta_A(\gamma*\delta), \beta_A(\delta)\}$, then by taking $t' = \frac{1}{2}\Big[\beta_A(\gamma*\delta) + \max\{\beta_A(\gamma), \beta_A(\delta)\}\Big]$, it follows that $\gamma*\delta \in U(\beta_A : t')$ and $\delta \in L(\beta_A : t')$, but $\gamma \notin L(\beta_A : t')$, which is a contradiction. Hence, $L(\beta_A : t')$ is not closed ideal of X.

Hence, $A = (\alpha_A, \beta_A)$ is an IFC-ideal of X since it satisfies (IBS3) and (IBS4). □

Theorem 3.21 *Let $S_1 \supseteq S_2 \supseteq S_3 \cdots$ be a descending chain of closed ideals of X which terminates at finite step. For an IFC-ideal $A = (\alpha_A, \beta_A)$ of X, if a sequence of elements of $Im(\alpha_A)$ is strictly increasing and $Im(\beta_A)$ strictly decreasing, then $A = (\alpha_A, \beta_A)$ is finite valued.*

Proof: Assume that $A = (\alpha_A, \beta_A)$ is infinite valued. Let $\{\psi_n\}$ be a strictly increasing sequence of elements of $Im(\alpha_A)$. Then $0 \leq \psi_1 < \psi_2 < \cdots < 1$. Note that $U(\alpha_A : \psi_t)$ is a closed ideal of X for $t = 1, 2, 3, \ldots$. Let $x \in U(\alpha_A : \psi_t)$ for $t = 2, 3, \ldots$. Then $\alpha_A(x) \geq \psi_t > \psi_{t-1}$, which implies that $x \in U(\alpha_A : \psi_{t-1})$. Hence $U(\alpha_A : \psi_t) \subseteq U(\alpha_A : \psi_{t-1})$ for $t = 2, 3, \ldots$. Since $\psi_{t-1} \in Im(\alpha_A)$ there exists x_{t-1} such that $\alpha_A(x_{t-1}) = \psi_{t-1}$. It follows that $x_{t-1} \in U(\alpha_A : \psi_{t-1})$, but $x_{t-1} \notin U(\alpha_A : \psi_t)$. Thus $U(\alpha_A : \psi_t) \subsetneq U(\alpha_A : \psi_{t-1})$, and so we obtain a strictly descending chain $U(\alpha_A : \psi_1) \supsetneq U(\alpha_A : \psi_2) \supsetneq \cdots$ of closed ideals of X which is not terminating. This is impossible.

Again let $\{\lambda_n\}$ be a strictly decreasing sequence of elements of $Im(\beta_A)$. Then $1 \geq \lambda_1 > \lambda_2 > \cdots > 0$. Note that $L(\beta_A : \lambda_s)$ is a closed ideal of X for $s = 1, 2, 3, \ldots$. Let $x \in L(\beta_A : \lambda_s)$ for $s = 2, 3, \ldots$. Then $\beta_A(x) \leq \lambda_s < \lambda_{s-1}$, which implies that $x \in L(\beta_A : \lambda_{s-1})$. Hence $L(\beta_A : \lambda_s) \subseteq L(\beta_A : \lambda_{s-1})$ for $s = 2, 3, \ldots$. Since $\lambda_{s-1} \in Im(\beta_A)$ there exists x_{s-1} such that $\beta_A(x_{s-1}) = \lambda_{s-1}$. It follows that $x_{s-1} \in L(\beta_A : \lambda_{s-1})$, but $x_{s-1} \notin L(\beta_A : \lambda_s)$. Thus $L(\beta_A : \lambda_s) \subsetneq L(\beta_A : \lambda_{s-1})$, and so we obtain a strictly descending chain $L(\beta_A : \lambda_1) \supsetneq L(\beta_A : \lambda_2) \supsetneq \cdots$ of closed ideals of X which is not terminating. This is impossible. Therefore, $A = (\alpha_A, \beta_A)$ is finite valued. □

Now we consider the converse of Theorem 3.21.

Theorem 3.22 *If every IFC-ideal A of X has the finite image, then every descending chain of closed ideals of X terminates at finite step.*

Proof: Suppose there exists a strictly descending chain $S_0 \supsetneq S_1 \supsetneq S_2 \cdots$ of closed ideals of X which does not terminate at finite step. Define an IFS $A = (\alpha_A, \beta_A)$ in X by

$$\alpha_A(x) = \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in S_n \setminus S_{n+1} \\ 1 & \text{if} \quad x \in \cap_{n=0}^{\infty} S_n \end{cases} \quad \text{and} \quad \beta_{\mathrm{A}}(\mathrm{x}) = \begin{cases} \frac{1}{n+1} & \text{if} \quad x \in S_n \setminus S_{n+1} \\ 0 & \text{if} \quad x \in \cap_{n=0}^{\infty} S_n \end{cases}$$

where $n = 0, 1, 2, \ldots$ and S_0 stands for X. Clearly, $\alpha_A(0 * x) \geq \alpha_A(x)$ and $\beta_A(0 * x) \leq \beta_A(x)$ for all $x \in X$. Let $x, y \in X$. Assume that $x * y \in S_n \setminus S_{n+1}$ and $y \in S_k \setminus S_{k+1}$ for $n = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $x * y$ and $y \in S_n$, so $x \in S_n$ because S_n is a closed ideal of X. Hence,

$$\alpha_A(x) \geq \frac{n}{n+1} = \min\{\alpha_A(x * y), \alpha_A(y)\}$$
$$\beta_A(x) \leq \frac{1}{n+1} = \max\{\beta_A(x * y), \beta_A(y)\}.$$

If $x*y, y \in \cap_{n=0}^{\infty} S_n$, then $x \in \cap_{n=0}^{\infty} S_n$. Thus

$$\alpha_A(x) = 1 = \min\{\alpha_A(x*y), \alpha_A(y)\}$$
$$\beta_A(x) = 0 = \max\{\beta_A(x*y), \beta_A(y)\}.$$

If $x*y \notin \cap_{n=0}^{\infty} S_n$ and $y \in \cap_{n=0}^{\infty} S_n$, then there exists a positive integer r such that $x*y \in S_r \setminus S_{r+1}$. It follows that $x \in S_r$ so that

$$\alpha_A(x) \geq \frac{r}{r+1} = \min\{\alpha_A(x*y), \alpha_A(y)\}$$
$$\beta_A(x) \leq \frac{1}{r+1} = \max\{\beta_A(x*y), \beta_A(y)\}.$$

Finally suppose that $x*y \in \cap_{n=0}^{\infty} S_n$ and $y \notin \cap_{n=0}^{\infty} S_n$. Then $y \in S_s \setminus S_{s+1}$ for some positive integer s. It follows that $x \in S_s$, and hence

$$\alpha_A(x) \geq \frac{s}{s+1} = \min\{\alpha_A(x*y), \alpha_A(y)\}$$
$$\beta_A(x) \leq \frac{1}{s+1} = \max\{\beta_A(x*y), \beta_A(y)\}.$$

This proves that $A = (\alpha_A, \beta_A)$ is an IFC-ideal with an infinite number of different values, which is a contradiction. This completes the proof. □

Theorem 3.23 *Every ascending chain of closed ideals of X terminates at finite step iff the set of values of any IFC-ideals is a well ordered subset of L.*

Proof: Let A be an intuitionistic fuzzy closed ideals of X. Suppose that the set of values of A is not a well-ordered subset of L. Then there exist a strictly decreasing sequence $\{\gamma_n\}$ such that $\alpha_A(x_n) = \gamma_n$. It follows that $U(\alpha_A : \gamma_1) \subsetneq U(\alpha_A : \gamma_2) \subsetneq U(\alpha_A : \gamma_3) \subsetneq \cdots$ is a strictly ascending chain of closed ideals of X which is not terminating. This is impossible.

If there exist a strictly increasing sequence $\{\delta_n\}$ such that $\beta_A(x_n) = \delta_n$. It follows that $L(\beta_A : \delta_1) \subsetneq L(\beta_A : \delta_2) \subsetneq L(\beta_A : \delta_3) \subsetneq \cdots$ is a strictly ascending chain of closed ideals of X which is not terminating. This is impossible.

To prove the converse suppose that there exist a strictly ascending chain

$$S_1 \subsetneq S_2 \subsetneq S_3 \subsetneq \cdots \tag{1}$$

of closed ideal of X which does not terminate at finite step. Note that $S := \cup_{n\in\mathbb{N}} S_n$ is a closed ideal of X. Define an IFS $A = (\alpha_A, \beta_A)$ in X by

$$\alpha_A(x) = \begin{cases} \frac{1}{k} & \text{where} \quad k = \min\{n \in \mathbb{N} | x \in S_n\} \\ 0 & \text{if} \quad x \notin S_n \end{cases} \quad \text{and} \quad \beta_A(x) = 1 - \alpha_A(x).$$

We claim that $A = (\alpha_A, \beta_A)$ is an IFC-ideal of X. Let $x \in X$. If $x \notin S_n$, then obviously $\alpha_A(0*x) \geq 0 = \alpha_A(x)$ and $\beta_A(0*x) \geq 1 = \beta_A(x)$. If $x \in$

$S_n \setminus S_{n-1}$ for $n = 2, 3, \ldots$, then $0 * x \in S_n$. Hence, $\alpha_A(0 * x) \geq \frac{1}{n} = \alpha_A(x)$ and $\beta_A(0*x) \geq 1 - \frac{1}{n} = \beta_A(x)$. Let $x, y \in X$. If $x*y \in S_n \setminus S_{n-1}$ and $y \in S_n \setminus S_{n-1}$ for $n = 2, 3, \ldots$ then $x \in S_n$. It follows that

$$\begin{aligned}\alpha_A(x) &\geq \frac{1}{n} = \min\{\alpha_A(x * y), \alpha_A(y)\}\\ \beta_A(x) &\leq 1 - \frac{1}{n} = \max\{\beta_A(x * y), \beta_A(y)\}.\end{aligned}$$

Suppose that $x * y \in S_n$ and $y \in S_n \setminus S_p$ for all $p < n$. Since $A = (\alpha_A, \beta_A)$ is a closed ideal of X, then $x \in S_n$, and so $\alpha_A(x) \geq \frac{1}{n} \geq \frac{1}{p+1} \geq \alpha_A(y)$ and $\beta_A(x) \leq 1 - \frac{1}{n} \leq 1 - \frac{1}{p+1} \leq \beta_A(y)$ and hence $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$, $\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\}$.

Similarly, for the case $x * y \in S_n \setminus S_p$ and $y \in S_n$, we have $\alpha_A(x) \geq \min\{\alpha_A(x * y), \alpha_A(y)\}$ and $\beta_A(x) \leq \max\{\beta_A(x * y), \beta_A(y)\}$.

Therefore $A = (\alpha_A, \beta_A)$ is an IFC-ideal of X. Since the chain (1) is not terminating, A has a strictly descending sequence of values. This contradicts that the value set of any IFC-ideal is well-ordered. This completes the proof. □

4 Investigation of intuitionistic fuzzy ideals under homomorphisms

In this section, homomorphism of intuitionistic fuzzy BG-algebra is defined and some results are studied.

Let f be a mapping from the set X into the set Y. Let B be an IFS in Y. Then the inverse image of B, is defined as $f^{-1}(B) = (f^{-1}(\alpha_B), f^{-1}(\beta_B))$ with the membership function and non-membership function respectively are given by $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$ and $f^{-1}(\beta_B)(x) = \beta_B(f(x))$. It can be shown that $f^{-1}(B)$ is an IFS.

Definition 4.1 *A mapping $f : X \to Y$ of BG-algebra is called a BG-homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Note that if $f : X \to Y$ is a BG-homomorphism, then $f(0) = 0$.*

Theorem 4.2 *Let $f : X \to Y$ be a homomorphism of BG-algebras. If $B = (\alpha_B, \beta_B)$ is an intuitionistic fuzzy ideal of Y, then the preimage $f^{-1}(B) = (f^{-1}(\alpha_B), f^{-1}(\beta_B))$ of B under f in X is an intuitionistic fuzzy ideal of X.*

Proof: For all $x \in X$, $f^{-1}(\alpha_B)(x) = \alpha_B(f(x)) \leq \alpha_B(0) = \alpha_B(f(0)) = f^{-1}(\alpha_B)(0)$ and $f^{-1}(\beta_B)(x) = \beta_B(f(x)) \geq \beta_B(0) = \beta_B(f(0)) = f^{-1}(\beta_B)(0)$. Let $x, y \in X$. Then $f^{-1}(\alpha_B)(x) = \alpha_B(f(x)) \geq \min\{\alpha_B((f(x)*f(y)), \alpha_B(f(y))\}$

$\geq \min\{\alpha_B(f(x*y), \alpha_B(f(y))\} = \min\{f^{-1}(\alpha_B)(x*y), f^{-1}(\alpha_B)(y)\}$ and $f^{-1}(\beta_B)(x) = \beta_B(f(x)) \leq \max\{\beta_B((f(x)*f(y)), \beta_B(f(y))\} \leq \max\{\beta_B(f(x*y), \beta_B(f(y)))\} = \max\{f^{-1}(\beta_B)(x*y), f^{-1}(\beta_B)(y)\}$. Hence, $f^{-1}(B) = (f^{-1}(\alpha_B), f^{-1}(\beta_B))$ is an intuitionistic fuzzy ideal of X. □

Theorem 4.3 *Let $f : X \to Y$ be an epimorphism of BG-algebras. Then $B = (\alpha_B, \beta_B)$ is an intuitionistic fuzzy ideal of Y, if $f^{-1}(B) = (f^{-1}(\alpha_B), f^{-1}(\beta_B))$ of B under f in X is an intuitionistic fuzzy ideal of X.*

Proof: For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then $\alpha_B(x) = \alpha_B(f(a)) = f^{-1}(\alpha_B)(a) \leq f^{-1}(\alpha_B)(0) = \alpha_B(f(0)) = \alpha_B(0)$ and $\beta_B(x) = \beta_B(f(a)) = f^{-1}(\beta_B)(a) \geq f^{-1}(\beta_B)(0) = \beta_B(f(0)) = \beta_B(0)$.

Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. Thus $\alpha_B(x) = \alpha_B(f(a)) = f^{-1}(\alpha_B)(a) \geq \min\{f^{-1}(\alpha_B)(a*b), f^{-1}(\alpha_B)(b)\} = \min\{\alpha_B(f(a*b)), \alpha_B(f(b))\} = \min\{\alpha_B(f(a)*f(b)), \alpha_B(f(b))\} = \min\{\alpha_B(x*y), \alpha_B(y)\}$ and $\beta_B(x) = \beta_B(f(a)) = f^{-1}(\beta_B)(a) \leq \max\{f^{-1}(\beta_B)(a*b), f^{-1}(\beta_B)(b)\} = \max\{\beta_B(f(a*b)), \beta_B(f(b))\} = \max\{\beta_B(f(a)*f(b)), \beta_B(f(b))\} = \max\{\beta_B(x*y), \beta_B(y)\}$. Then $B = (\alpha_B, \beta_B)$ is an intuitionistic fuzzy ideal of Y. □

5 Equivalence relations on intuitionistic fuzzy ideals

Let IFI(X) denote the family of all intuitionistic fuzzy ideals of X and let $\rho \in [0,1]$. Define binary relations U^ρ and L^ρ on IFI(X) as $(A, B) \in U^\rho \Leftrightarrow U(\alpha_A : \rho) = U(\alpha_B : \rho)$ and $(A, B) \in L^\rho \Leftrightarrow L(\beta_A : \rho) = L(\beta_B : \rho)$ respectively, for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in IFI(X). Then clearly U^ρ and L^ρ are equivalence relations on IFI(X). For any $A = (\alpha_A, \beta_A) \in IFI(X)$, let $[A]_{U^\rho}$ (respectively, $[A]_{L^\rho}$) denote the equivalence class of A modulo U^ρ (respectively, L^ρ), and denote by IFI(X)/U^ρ (respectively, IFI(X)/L^ρ) the collection of all equivalence classes modulo U^ρ (respectively, L^ρ), i.e.,

$$\text{IFI(X)}/U^\rho := \{[A]_{U^\rho} | A = (\alpha_A, \beta_A) \in IFI(X)\},$$

respectively,

$$\text{IFI(X)}/L^\rho := \{[A]_{L^\rho} | A = (\alpha_A, \beta_A) \in IFI(X)\}.$$

These two sets are also called the quotient sets.

Now let $T(X)$ denote the family of all ideals of X and let $\rho \in [0,1]$. Define mappings f_ρ and g_ρ from IFI(X) to $T(X) \cup \{\phi\}$ by $f_\rho(A) = U(\alpha_A : \rho)$ and $g_\rho(A) = L(\beta_A : \rho)$, respectively, for all $A = (\alpha_A, \beta_A) \in IFI(X)$. Then f_ρ and g_ρ are clearly well-defined.

Theorem 5.1 *For any $\rho \in [0,1]$, the maps f_ρ and g_ρ are surjective from IFI(X) to $T(X) \cup \{\phi\}$.*

Proof: Let $\rho \in [0,1]$. Note that $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1})$ is in IFI(X), where $\mathbf{0}$ and $\mathbf{1}$ are fuzzy sets in X defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}\ (x) = 1$ for all $x \in X$. Obviously $f_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) = \phi = L(\mathbf{1} : \rho) = g_\rho(\mathbf{0}_\sim)$. Let $P(\neq \phi) \in IFI(X)$. For $P_\sim = (\chi_P, \overline{\chi}_P) \in IFI(X)$, we have $f_\rho(P_\sim) = U(\chi_P : \rho) = P$ and $g_\rho(P_\sim) = L(\overline{\chi}_P : \rho) = P$. Hence f_ρ and g_ρ are surjective. □

Theorem 5.2 *The quotient sets IFI(X)/U^ρ and IFI(X)/L^ρ are equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in [0,1]$.*

Proof: For $\rho \in [0,1]$ let f^*_ρ (respectively, g^*_ρ) be a map from IFI(X)/U^ρ (respectively, IFI(X)/L^ρ) to $T(X) \cup \{\phi\}$ defined by $f^*_\rho([A]_{U^\rho}) = f_\rho(A)$ (respectively, $g^*_\rho([A]_{U^\rho}) = g_\rho(A)$) for all $A = (\alpha_A, \beta_A) \in IFI(X)\}$. If $U(\alpha_A : \rho) = U(\alpha_B : \rho)$ and $L(\beta_A : \rho) = L(\beta_B : \rho)$ for $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ in IFI(X), then $(A, B) \in U^\rho$ and $(A, B) \in L^\rho$; hence $[A]_{U^\rho} = [B]_{U^\rho}$ and $[A]_{L^\rho} = [B]_{L^\rho}$. Therefore the maps f^*_ρ and g^*_ρ are injective. Now let $P(\neq \phi) \in IFI(X)$. For $P_\sim = (\chi_P, \overline{\chi}_P) \in IFI(X)$, we have

$$f^*_\rho([P_\sim]_{U^\rho}) = f_\rho(P_\sim) = U(\chi_P : \rho) = P,$$

and

$$g^*_\rho([P_\sim]_{L^\rho}) = g_\rho(P_\sim) = L(\overline{\chi}_P : \rho) = P.$$

Finally, for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFI(X)$ we get

$$f^*_\rho([\mathbf{0}_\sim]_{U^\rho}) = f_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) = \phi$$

and

$$g^*_\rho([\mathbf{0}_\sim]_{L^\rho}) = g_\rho(\mathbf{0}_\sim) = L(\mathbf{1} : \rho) = \phi.$$

This shows that f^*_ρ and g^*_ρ are surjective. This completes the proof. □

For any $\rho \in [0,1]$, we define another relation R^ρ on IFI(X) as follows:

$$(A, B) \in R^\rho \Leftrightarrow U(\alpha_A : \rho) \cap L(\beta_A : \rho) = U(\alpha_B : \rho) \cap L(\beta_B : \rho)$$

for any $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B) \in IFI(X)$. Then the relation R^ρ is an equivalence relation on IFI(X).

Theorem 5.3 *For any $\rho \in [0,1]$, the maps $\psi_\rho : IFI(X) \to T(X) \cap \{\phi\}$ defined by $\psi_\rho(A) = f_\rho(A) \cap g_\rho(A)$ for each $A = (\alpha_A, \beta_A) \in X$ is surjective.*

Proof: Let $\rho \in [0,1]$. For $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFI(X)$,

$$\psi_\rho(\mathbf{0}_\sim) = f_\rho(\mathbf{0}_\sim) \cap g_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) \cap L(\mathbf{1} : \rho) = \phi.$$

For any $H \in IFI(X)$, there exists $H_\sim = (\chi_H, \overline{\chi}_H) \in IFI(X)$ such that

$$\psi_\rho(H_\sim) = f_\rho(H_\sim) \cap g_\rho(H_\sim) = U(\chi_H : \rho) \cap L(\overline{\chi}_H : \rho) = H.$$

This completes the proof. □

Theorem 5.4 *The quotient sets IFI(X)/R^ρ are equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in [0,1]$.*

Proof: For $\rho \in [0,1]$, define a map $\psi^*_\rho : IFI(X)/R^\rho \to T(X) \cup \{\phi\}$ by

$$\psi^*_\rho([A]_{R^\rho}) = \psi_\rho(A)$$

for all $[A]_{R^\rho} \in IFI(X)/R^\rho$. Assume that $\psi^*_\rho([A]_{R^\rho}) = \psi^*_\rho([B]_{R^\rho})$ for any $[A]_{R^\rho}$ and $[B]_{R^\rho} \in IFI(X)/R^\rho$. Then $f_\rho(A) \cap g_\rho(A) = f_\rho(B) \cap g_\rho(B)$, i.e.,

$$U(\alpha_A : \rho) \cap L(\beta_A : \rho) = U(\alpha_B : \rho) \cap L(\beta_B : \rho).$$

Hence $(A, B) \in R^\rho$, and so $[A]_{R^\rho} = [B]_{R^\rho}$. Therefore the maps ψ^*_ρ are injective. Now for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFI(X)$ we have

$$\psi^*_\rho([\mathbf{0}_\sim]_{R^\rho}) = \psi_\rho(\mathbf{0}_\sim) = f_\rho(\mathbf{0}_\sim) \cap g_\rho(\mathbf{0}_\sim) = U(\mathbf{0} : \rho) \cap L(\mathbf{1} : \rho) = \phi.$$

If $H \in IFI(X)$, then for $H_\sim = (\chi_H, \overline{\chi}_H) \in IFI(X)$, we obtain

$$\psi^*_\rho([H_\sim]_{R^\rho}) = \psi_\rho(H_\sim) = f_\rho(H_\sim) \cap g_\rho(H_\sim) = U(\chi_H : \rho) \cap L(\overline{\chi}_H : \rho) = H.$$

Thus ψ^*_ρ is surjective. This completes the proof. □

6 Product of intuitionistic fuzzy BG-algebras

In this section, product of intuitionistic fuzzy ideals in BG-algebra is defined and some results are studied.

Definition 6.1 *Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two IFSs of X. The cartesian product $A \times B = (\alpha_A \times \alpha_B, \beta_A \times \beta_B)$ is defined by*

$$(\alpha_A \times \alpha_B)(x, y) = \min\{\alpha_A(x), \alpha_B(y)\} \text{ and}$$
$$(\beta_A \times \beta_B)(x, y) = \max\{\beta_A(x), \beta_B(y)\},$$

where $\alpha_A \times \alpha_B : X \times X \to [0, 1]$ and $\beta_A \times \beta_B : X \times X \to [0, 1]$ for all $x, y \in X$.

Proposition 6.2 *Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be intuitionistic fuzzy ideals of X, then $A \times B$ is an intuitionistic fuzzy ideal of $X \times X$.*

Proof: For any $(x, y) \in X \times X$, we have $(\alpha_A \times \alpha_B)(0, 0) = \min\{\alpha_A(0), \alpha_B(0)\} \geq \min\{\alpha_A(x), \alpha_B(y)\} = (\alpha_A \times \alpha_B)(x, y)$ and $(\beta_A \times \beta_B)(0, 0) = \max\{\beta_A(0), \beta_B(0)\} \leq \min\{\beta_A(x), \beta_B(y)\} = (\beta_A \times \beta_B)(x, y)$. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then $(\alpha_A \times \alpha_B)(x_1, y_2) = \min\{\alpha_A(x_1), \alpha_B(y_1)\} \geq \min\{\min\{\alpha_A(x_1 * x_2), \alpha_A(x_2)\}, \min\{\alpha_B(y_1 * y_2), \alpha_B(y_2)\}\} = \min\{\min\{\alpha_A(x_1 * x_2), \alpha_B(y_1 * y_2)\}, \min\{\alpha_A(x_2), \alpha_B(y_2)\}\} = \min\{(\alpha_A \times \alpha_B)(x_1 * x_2, y_1 * y_2), (\alpha_A \times \alpha_B)(x_2, y_2)\} = \min\{(\alpha_A \times \alpha_B)((x_1, y_1) * (x_2, y_2)), (\alpha_A \times \alpha_B)(x_2, y_2)\}$ and $(\beta_A \times \beta_B)(x_1, y_2) = \max\{\beta_A(x_1), \beta_B(y_1)\} \leq \max\{\max\{\beta_A(x_1 * x_2), \beta_A(x_2)\}, \max\{\beta_B(y_1 * y_2), \beta_B(y_2)\}\} = \max\{\max\{\beta_A(x_1 * x_2), \beta_B(y_1 * y_2)\}, \max\{\beta_A(x_2), \beta_B(y_2)\}\} = \max\{(\beta_A \times \beta_B)(x_1 * x_2, y_1 * y_2), (\beta_A \times \beta_B)(x_2, y_2)\} = \max\{(\beta_A \times \beta_B)((x_1, y_1) * (x_2, y_2)), (\beta_A \times \beta_B)(x_2, y_2)\}$. Hence, $A \times B$ is an intuitionistic fuzzy ideal of $X \times X$. □

Proposition 6.3 *Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are IFC-ideals of X, then $A \times B$ is an IFC-ideal of $X \times X$.*

Proof: Now, $(\alpha_A \times \alpha_B)((0, 0) * (x, y)) = (\alpha_A \times \alpha_B)(0 * x, 0 * y) = \min\{\alpha_A(0 * x), \alpha_B(0 * y)\} \geq \min\{\alpha_A(x), \alpha_B(y)\} = (\alpha_A \times \alpha_B)(x, y)$ and $(\beta_A \times \beta_B)((0, 0) * (x, y)) = (\beta_A \times \beta_B)(0 * x, 0 * y) = \max\{\beta_A(0 * x), \beta_B(0 * y)\} \leq \max\{\beta_A(x), \beta_B(y)\} = (\beta_A \times \beta_B)(x, y)$. Hence, $A \times B$ is an IFC-ideal of $X \times X$. □

Lemma 6.4 *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy ideals of X, then $\bigoplus(A \times B) = (\alpha_A \times \alpha_B, \overline{\alpha}_A \times \overline{\alpha}_B)$ is an intuitionistic fuzzy ideals of $X \times X$.*

Proof: Now, $(\alpha_A \times \alpha_B)(x, y) = \min\{\alpha_A(x), \alpha_B(y)\} \Rightarrow 1-(\overline{\alpha}_A \times \overline{\alpha}_B)(x, y) = \min\{1-\overline{\alpha}_A(x), 1-\overline{\alpha}_B(y)\} \Rightarrow 1-\min\{1-\overline{\alpha}_A(x), 1-\overline{\alpha}_B(y)\} = (\overline{\alpha}_A \times \overline{\alpha}_B)(x, y) \Rightarrow (\overline{\alpha}_A \times \overline{\alpha}_B)(x, y) = \max\{\overline{\alpha}_A(x), \overline{\alpha}_B(y)\}$. Hence, $\bigoplus(A \times B)$ is an intuitionistic fuzzy ideal of $X \times X$. □

Lemma 6.5 *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy ideals of X, then $\bigotimes(A \times B) = (\overline{\beta}_A \times \overline{\beta}_B, \beta_A \times \beta_B)$ is an intuitionistic fuzzy ideal of $X \times X$.*

Proof: Since $(\beta_A \times \beta_B)(x, y) = \max\{\beta_A(x), \beta_B(y)\}$. That implies, $1-(\overline{\beta}_A \times \overline{\beta}_B)(x, y) = \max\{1-\overline{\beta}_A(x), 1-\overline{\beta}_B(y)\}$. This is, $1-\max\{1-\overline{\beta}_A(x), 1-\overline{\beta}_B(y)\} = (\overline{\beta}_A \times \overline{\beta}_B)(x, y)$. Therefore, $(\overline{\beta}_A \times \overline{\beta}_B)(x, y) = \min\{\overline{\beta}_A(x), \overline{\beta}_B(y)\}$. Hence, $\bigotimes(A \times B) = (\overline{\beta}_A \times \overline{\beta}_B, \beta_A \times \beta_B)$ is an intuitionistic fuzzy ideal of $X \times X$. □

By the above two Lemmas, it is not difficult to verify that the following theorem is valid.

Theorem 6.6 *The IFSs $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy ideals of X if and only if $\bigoplus(A \times B) = (\alpha_A \times \alpha_B, \overline{\alpha}_A \times \overline{\alpha}_B)$ and $\bigotimes(A \times B) = (\overline{\beta}_A \times \overline{\beta}_B, \beta_A \times \beta_B)$ are intuitionistic fuzzy ideal of $X \times X$.*

Lemma 6.7 *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are IFC-ideals of X, then $\bigoplus(A \times B) = (\alpha_A \times \alpha_B, \overline{\alpha}_A \times \overline{\alpha}_B)$ is an IFC-ideals of $X \times X$.*

Proof: Since $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are IFC-ideals of X, $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are intuitionistic fuzzy ideals of X. Thus, $A \times B$ is intuitionistic fuzzy ideal of $X \times X$.

Now $(\alpha_A \times \alpha_B)((0, 0) * (x, y)) \geq (\alpha_A \times \alpha_B)(x, y)$. That is, $1 - (\overline{\alpha}_A \times \overline{\alpha}_B)((0, 0) * (x, y)) \geq 1 - (\overline{\alpha}_A \times \overline{\alpha}_B)(x, y)$. This gives, $(\overline{\alpha}_A \times \overline{\alpha}_B)((0, 0) * (x, y)) \leq (\overline{\alpha}_A \times \overline{\alpha}_B)(x, y)$. Hence, $\bigoplus(A \times B) = (\alpha_A \times \alpha_B, \overline{\alpha}_A \times \overline{\alpha}_B)$ is an IFC-ideal of $X \times X$. □

Lemma 6.8 *If $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are IFC-ideals of X, then $\bigotimes(A \times B) = (\overline{\beta}_A \times \overline{\beta}_B, \beta_A \times \beta_B)$ is an IFC-ideals of $X \times X$.*

Proof: The proof is similar to the proof of the above Lemma. □

The following theorem follows from the above two Lemmas.

Theorem 6.9 *$A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are IFC-ideals of X iff $\bigoplus(A \times B) = (\alpha_A \times \alpha_B, \overline{\alpha}_A \times \overline{\alpha}_B)$ and $\bigotimes(A \times B) = (\overline{\beta}_A \times \overline{\beta}_B, \beta_A \times \beta_B)$ are IFC-ideal of $X \times X$.*

Definition 6.10 *Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ is intuitionistic fuzzy ideals of X. For $s, t \in [0,1]$, the set $U(\alpha_A \times \alpha_B : s) = \{(x,y) \in X \times X | (\alpha_A \times \alpha_B)(x,y) \geq s\}$ is called upper s-level of $A \times B$ and $L(\beta_A \times \beta_B : t) = \{(x,y) \in X \times X | (\beta_A \times \beta_B)(x,y) \leq t\}$ is called lower t-level of $A \times B$.*

Theorem 6.11 *For any IFS $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$, $A \times B$ is an IFC-ideals of $X \times X$ if and only if the non-empty upper s-level cut $U(\alpha_A \times \alpha_B : s)$ and the non-empty lower t-level cut $L(\beta_A \times \beta_B : t)$ are closed ideals of $X \times X$ for any s and $t \in [0,1]$.*

Proof: Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ are IFC-ideals of X, therefore for any $(x,y) \in X \times X$, $(\alpha_A \times \alpha_B)((0,0) * (x,y)) \geq (\alpha_A \times \alpha_B)(x,y)$ and $(\beta_A \times \beta_B)((0,0) * (x,y)) \leq (\beta_A \times \beta_B)(x,y)$. For $s \in [0,1]$, if $(\alpha_A \times \alpha_B)(x,y) \geq s$. That is, $(\alpha_A \times \alpha_B)((0,0) * (x,y)) \geq s$. This implies, $(0,0) * (x,y) \in U(\alpha_A \times \alpha_B : s)$.

Let $(x,y), (x',y') \in X \times X$ such that $(x,y) * (x',y') \in U(\alpha_A \times \alpha_B : s)$ and $(x',y') \in U(\alpha_A \times \alpha_B : s)$. Then, $(\alpha_A \times \alpha_B)(x,y) \geq \min\{(\alpha_A \times \alpha_B)((x,y) * (x',y')), (\alpha_A \times \alpha_B)(x',y')\} \geq \min(s,s) = s$. This implies, $(x,y) \in U(\alpha_A \times \alpha_B : s)$. Thus $U(\alpha_A \times \alpha_B : s)$ is closed ideal of $X \times X$. Similarly, $L(\beta_A \times \beta_B : t)$ is closed ideal of $X \times X$.

Conversely, let $(x,y) \in X \times X$ such that $(\alpha_A \times \alpha_B)(x,y) = s$ and $(\beta_A \times \beta_B)(x,y) = t$. This implies, $(x,y) \in U(\alpha_A \times \alpha_B : s)$ and $(x,y) \in L(\beta_A \times \beta_B : t)$. Since $(0,0) * (x,y) \in U(\alpha_A \times \alpha_B : s)$ and $(0,0) * (x,y) \in L(\beta_A \times \beta_B : t)$ (by definition of closed ideal). Therefore, $(\alpha_A \times \alpha_B)((0,0) * (x,y)) \geq s$ and $(\beta_A \times \beta_B)((0,0) * (x,y)) \leq t$. This gives, $(\alpha_A \times \alpha_B)((0,0) * (x,y)) \geq (\alpha_A \times \alpha_B)(x,y)$ and $(\beta_A \times \beta_B)((0,0) * (x,y)) \leq (\beta_A \times \beta_B)(x,y)$. Hence, $A \times B$ is an IFC-ideal of $X \times X$. □

7 Conclusions and future work

In the present paper, we have presented some extended results of intuitionistic fuzzy ideal called intuitionistic fuzzy closed ideals of BG-algebras and investigated some of their useful properties. The product of BG-subalgebra has been introduced and some important properties are of it are also studied. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as KK-algebras, lattices and Lie algebras.

It is our hope that this work would other foundations for further study of the theory of BG-algebras. In our future study of fuzzy structure of BG-algebra, may be the following topics should be considered: (i) To find intuitionistic (T,S)-fuzzy closed ideals of BG-algebras, where S and T are given imaginable triangular norms, (ii) To get more results in intuitionistic fuzzy closed ideals of BG-algebra and application, (iii) To find $(\epsilon, \epsilon \vee q)$- intuitionistic fuzzy ideals of BG-algebras.

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References

[1] S.S. Ahn and H.D. Lee, Fuzzy subalgebras of BG-algebras, *Commun. Korean Math. Soc.* 19(2) (2004), 243-251.

[2] K.T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986), 87-96.

[3] K.T. Atanassov, More on intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 33(1) (1989), 37-46.

[4] K.T. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 61 (1994), 137-142.

[5] J.R. Cho and H.S. Kim, On B-algebras and quasigroups, *Quasigroups and Related Systems* 7 (2001), 1-6.

[6] Q.P. Hu and X. Li, On BCH-algebras, *Math. Seminar Notes* 11 (1983), 313-320.

[7] Y. Imai and K. Iseki, On axiom system of propositional calculi, *XIV Proc. Japan Academy*, 42 (1966), 19-22.

[8] K. Iseki and S. Tanaka, An introduction to the theory of BCK-algebras, *Math. Japonica*, 23 (1978), 1-26.

[9] C.B. Kim and H.S. Kim, On BG-algebras, *Demonstratio Mathematica*, 41 (2008), 497-505.

[10] R. Muthuraj, M. Sridharan and P.M. Sitharselvam, Fuzzy BG-ideals in BG-algebra, *Int. J. of Computer Applications*, 2(1) (2010), 26-30.

[11] J. Neggers and H.S. Kim, On B-algebras, *Math. Vensik* 54 (2002), 21-29.

[12] H.K. Park and H.S. Kim, On quadratic B-algebras, *Quasigroups and Related Systems* 7 (2001), 6772.

[13] A.B. Saeid, Fuzzy topological BG-algbra, *Int. Math. J.*, 6(2) (2005), 121-127.

[14] T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy closed ideals of BG-algebra and their products, *International Journal of Fuzzy Logic Systems*, 2(2) (2012), 27-44.

[15] T. Senapati, M. Bhowmik and M. Pal, On intuitionistic fuzzy subalgebras in BG-algebras (Communicated).

[16] T. Senapati, M. Bhowmik and M. Pal, Fuzzy B-subalgebras of B-algebra with respect t-norm, *Journal of Fuzzy Set Valued Analysis*, Volume-2012 (2012), 11-pages.

[17] T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy BG-subalgebras, *The J. of Fuzzy Mathematics*, 20(3) (2012), 707-720.

[18] L.A. Zadeh, Fuzzy sets, *Inform. and Control*, 8 (1965), 338-353.

[19] L.A. Zadeh, Toward a generalized theory of uncertainty (GTU)-an outline, *Inform. Sci.*, 172 (2005), 1-40.

[20] A. Zarandi and A.B. Saeid, Intuitionistic fuzzy ideals of BG-algebras, *World Academy of Science, Engineering and Technology*, 5 (2005), 187-189.

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