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On approximate m-convexity of sub-homogeneous functions

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Abstract

In this article we introduce the concepts of approximate Jensen m-convexity and approximate Wright m-convexity for real valued functions defined on the set of nonnegative real numbers. We prove some Bernstein-Doetsch type results for real valued sub-homogeneous functions defined on the set of all positive real numbers.

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1 Introduction

We start by recalling a couple of definitions: ε-m-convex and sub-homogeneous real valued functions defined on $[0,+\infty)$. Additionally, we introduce in the approximate fashion two new definitions for real valued functions, on the one hand, the ε-Jensen m-convex function, and on the other hand, the ε-Wright m-convex function assuming that all the functions considered are defined on $[0,+\infty)$. As it is customary, $m \in [0,1]$ and sometimes, either $m = 0$ or $m = 1$ will be discarded.

Definition 1.1 *([4]) Let $\varepsilon > 0$ and $m \in [0,1]$. A function $f : [0,+\infty) \to \mathbf{R}$ is called ε-m-convex if for any $x, y \in [0,+\infty)$ and $t \in [0,1]$*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + \varepsilon.$$

Definition 1.2 *([1]) A function $f : [0,+\infty) \to \mathbf{R}$ is called sub-homogeneous (or quasi-homogeneous [3]) if $f(\lambda x) \leq \lambda f(x)$ for all $\lambda \geq 1$ and $x \in [0,+\infty)$.*

By following ideas from [6], we introduce the concept of an approximately Jensen m-convex function.

Definition 1.3 *Let $\varepsilon > 0$ and $m \in (0,1]$. A function $f : [0,+\infty) \to \mathbf{R}$ is called ε-Jensen m-convex if for any $x, y \in [0,+\infty)$*

$$f\left(\frac{x+y}{c_m}\right) \leq \frac{f(x)+f(y)}{c_m} + \varepsilon, \tag{1}$$

where $c_m = 1 + \dfrac{1}{m}$.

The class of all ε-Jensen m-convex functions on $[0,+\infty)$ will be denoted as $J_{\varepsilon,m}[+\infty)$. In the same fashion, $J_{\varepsilon,m}(+\infty)$ will denote the class of all ε-Jensen m-convex functions on $(0,+\infty)$.

Definition 1.4 *Let $\varepsilon > 0$ and $m \in [0,1]$. A function $f : [0,+\infty) \to \mathbf{R}$ is called ε-Wright m-convex if for any $x, y \in [0,+\infty)$ and $t \in [0,1]$*

$$f(tx + m(1-t)y) + f(m(1-t)x + ty) \leq [t + m(1-t)][f(x) + f(y)] + 2\varepsilon. \tag{2}$$

We denote the class of all ε-Wright m-convex functions on $[0,+\infty)$ as $W_{\varepsilon,m}[+\infty)$ and the class of all ε-Wright m-convex functions on $(0,+\infty)$ as $W_{\varepsilon,m}(+\infty)$.

If we take $\varepsilon = 0$ in Definition 1.1, we obtain the usual notion of m-convexity ([4]); and if (1) and (2) hold with $\varepsilon = 0$, then f is called Jensen m-convex function ([5]) (denoted by J_m instead of $J_{\varepsilon,m}$) and Wright m-convex function (denoted by W_m instead of $W_{\varepsilon,m}$), respectively.

2 Examples and Counterexamples

In order to justify the definitions of the types of functions recently introduced, we present some results through which is possible to generate examples and counterexamples of them.

Example 2.1 *A bunch of examples (counterexamples) of sub-homogeneous (non sub-homogeneous) functions, respectively, can be constructed as follows. First of all, it is straightforward to prove the next proposition:*

Let $f : (0, +\infty) \to \mathbf{R}$ *be any real function. The following statements are equivalent:*

1. f *is sub-homogeneous on* $(0, +\infty)$.
2. $f(x)/x$ *is decreasing on* $(0, +\infty)$.

If additionally, f *is differentiable on* $(0, +\infty)$, *then we have the following equivalence:*

f *is sub-homogeneous on* $(0, +\infty)$ *iff* $h_f(x) = xf'(x) - f(x) \leq 0$ *for all* $x \in (0, +\infty)$.

Well, based on the last characterization we can prove the next result:

Proposition 2.2 *The cubic real polynomial function* $f(x) = ax^3 + bx^2 + cx + d$ *is sub-homogeneous on* $(0, +\infty)$ *iff the coefficients of* f *satisfy any of the following conditions:*

1. $a < 0, b \leq 0, d \geq 0$.
2. $a < 0, b > 0, d > 0$, *and* $h_f(-\frac{b}{3a}) \leq 0$.

For instance, the polynomial function $-x^3 + 1$ *is sub-homogeneous but* $-x^3 + 3x^2 + \dfrac{1}{2}$ *is not.*

Example 2.3 *An example of an* ε*-Jensen* m*-convex function which is not Jensen* m*-convex is the following:*

Given $\varepsilon > 0$, *set up* $m_1 = \dfrac{\sqrt{33} - 1}{16} \approx 0.296535$ *and* $g(m) = \dfrac{8m(1+m)}{27(1+3m)^2}$ *for all* $m \in (0, m_1]$. *It is not very difficult to verify the next statement:*

If $a > 0, b < 0, d < 0$ *and* $0 < g(m) - a^2b^{-3}d \leq -a^2b^{-3}(1+m)(1-m)^{-1}\varepsilon$, *then the real polynomial function* $f(x) = ax^3 + bx^2 + cx + d \in J_{\varepsilon,m}[+\infty) - J_m[+\infty)$.

For instance, if $0 < \varepsilon < \dfrac{1}{53}$, *then* $f(x) = 2x^3 - 2x^2 - g(m_\varepsilon) \in J_{\varepsilon,m_\varepsilon}[+\infty) - J_{m_\varepsilon}[+\infty)$, *where*

$$m_\varepsilon = \frac{8 - 159\,\varepsilon - 8\sqrt{1 - 53\,\varepsilon}}{16 + 477\varepsilon} \in (0, 0.2) \subset (0, m_1].$$

Remark 2.4 *Another way of producing an example of an ε-Jensen m-convex function is by noting that if $-\infty < a < b \leq \dfrac{2a}{c_m} + \varepsilon$, then any function $f : [0, +\infty) \to [a, b]$ is ε-Jensen m-convex.*

Example 2.5 *The function $f : (0, +\infty) \to \mathbf{R}$ given by $f(x) = e^{-x}$ is an example of a function which is sub-homogeneous and ε-Jensen m-convex (and ε-Wright m-convex) at the same time. In fact, because $f(x)/x$ is decreasing on $(0, +\infty)$, f is sub-homogeneous. Furthermore, by the convexity of f ([7]), we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + (1-t)[f(my) - mf(y)].$$

Since f is positive and decreasing, we obtain

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + f(my) - mf(y).$$

Moreover, $0 < f(x) < 1$ for all $x \in (0, +\infty)$. Thus, $f(my) - mf(y) < 1$ and hence

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + 1.$$

Therefore, f is 1-m-convex, which in turn implies that for $m \in (0, 1]$ $f \in J_{1,m}(+\infty)$ (as well $f \in W_{1,m}(+\infty)$).

3 Main Results

In this section, we present a list of results concerning to inequalities of Jensen type in the discrete case and local boundless. Based on these findings some other results of Bernstein-Doetsch type for sub-homogeneous functions are also obtained.

Theorem 3.1 *Let $\varepsilon > 0$ and $m \in (0, 1)$. If $f : (0, +\infty) \to \mathbf{R}$ is in $J_{\varepsilon,m}(+\infty)$, then f satisfies the following inequality*

$$f\left(\frac{1}{c_m^n}\sum_{i=1}^{2^n} x_i\right) \leq \frac{1}{c_m^n}\sum_{i=1}^{2^n} f(x_i) + \sum_{i=0}^{n-1}\left(\frac{2}{c_m}\right)^i \varepsilon, \tag{3}$$

for all $n \in \mathbf{N}$ and $x_1, \ldots, x_{2^n} \in (0, +\infty)$.

Proof. The proof is by induction on n. If $n = 1$, it is clear because $f \in J_{\varepsilon,m}(+\infty)$. We assume now that the result is true for n. But,

$$f\left(\frac{1}{c_m^{n+1}}\sum_{i=1}^{2^{n+1}} x_i\right) = f\left(\frac{1}{c_m^{n+1}}\sum_{i=1}^{2^n}[x_i + x_{2^n+i}]\right) = f\left(\frac{1}{c_m^n}\sum_{i=1}^{2^n}\frac{x_i + x_{2^n+i}}{c_m}\right).$$

Therefore, by using the inductive hypothesis and since $f \in J_{\varepsilon,m}(+\infty)$, we have

$$\begin{aligned} f\left(\frac{1}{c_m^{n+1}}\sum_{i=1}^{2^{n+1}} x_i\right) &\leq \frac{1}{c_m^n}\sum_{i=1}^{2^n} f\left(\frac{x_i + x_{2^n+i}}{c_m}\right) + \sum_{i=0}^{n-1}\left(\frac{2}{c_m}\right)^i \varepsilon \\ &\leq \frac{1}{c_m^n}\sum_{i=1}^{2^n}\left[\frac{f(x_i)+f(x_{2^n+i})}{c_m} + \varepsilon\right] + \sum_{i=0}^{n-1}\left(\frac{2}{c_m}\right)^i \varepsilon \\ &= \frac{1}{c_m^{n+1}}\sum_{i=1}^{2^{n+1}} f(x_i) + \sum_{i=0}^{n}\left(\frac{2}{c_m}\right)^i \varepsilon. \end{aligned}$$

Corollary 3.2 *On the same conditions of Theorem 3.1 the following inequality holds*

$$f\left(\frac{k}{2^n}x + m\left(1-\frac{k}{2^n}\right)y\right) \leq \frac{k}{c_m^n} f\left(\left[\frac{c_m}{2}\right]^n x\right) + m\left(\frac{2^n-k}{c_m^n}\right) f\left(\left[\frac{c_m}{2}\right]^n y\right) + \frac{(m+2)c_m\varepsilon}{c_m-2}$$

for all $x, y > 0$, $n \in \mathbf{N}$, *and* $k \in \{0,1,\ldots,2^n\}$.

Proof. It is not difficult to prove that if $f \in J_{\varepsilon,m}[+\infty)$, then $f(mx) \leq mf(x) + (m+1)\varepsilon$. By using this fact, taking $x_1 = \cdots = x_k = \left(\frac{c_m}{2}\right)^n x$ and $x_{k+1} = \cdots = x_{2^n} = m\left(\frac{c_m}{2}\right)^n y$ in (3), and taking into account that $\frac{2^n-k}{c_m^n} \leq 1$, the result is obtained.

Proposition 3.3 *Let* $f : (0,+\infty) \to \mathbf{R}$ *be a sub-homogeneous function in* $J_{\varepsilon,m}(+\infty)$, *with* $m \in (0,1)$. *If* f *is locally bounded from above at a point* $p \in (0,+\infty)$, *then* f *is locally bounded from above on* $(0,+\infty)$.

Proof. Let $U \subset (0,+\infty)$ be an open set such that $p \in U$ and $f(u) \leq M$ for all $u \in U$, with $M \in \mathbf{R}^+$. Let $q \in (0,+\infty)$ be arbitrary. Since

$$\lim_{n\to+\infty} \frac{q - 2^{-n}p}{m(1-2^{-n})} = \frac{q}{m} \in (0,+\infty),$$

there exists $n \in \mathbf{N}$ such that

$$y_0 := \frac{q - 2^{-n}p}{m(1-2^{-n})} \in (0,+\infty),$$

and we have $q = 2^{-n}p + m(1-2^{-n})y_0$. Let $V = W + m(1-2^{-n})y_0$, where $W = \{2^{-n}u : u \in U\}$. Then V is open and $q \in V$. Now, if $v \in V$, then $v = 2^{-n}u_0 + m(1-2^{-n})y_0$ for some $u_0 \in U$. Therefore, by Corollary 3.2,

$$f(v) \leq c_m^{-n} f\left(\left[\frac{c_m}{2}\right]^n u_0\right) + m\left(\frac{2^n-1}{c_m^n}\right) f\left(\left[\frac{c_m}{2}\right]^n y_0\right) + \frac{(m+2)c_m\varepsilon}{c_m-2}.$$

By using now the sub-homogeneity of f, we have

$$f(v) \leq 2^{-n}M + m(1-2^{-n})f(y_0) + \frac{(m+2)c_m\varepsilon}{c_m - 2}.$$

Thus f is locally bounded from above on $(0,+\infty)$.

Proposition 3.4 *Let $f : (0,+\infty) \to \mathbf{R}$ be a sub-homogeneous function in $J_{\varepsilon,m}(+\infty)$, with $m \in (0,1)$. If f is locally bounded from below at a point $p \in (0,+\infty)$, then f is locally bounded from below on $(0,+\infty)$.*

Proof. Let $U \subset (0,+\infty)$ be an open set such that $p \in U$ and $f(u) \geq M$ for all $u \in U$, with $M \in \mathbf{R}$. Let $q \in (0,+\infty)$ be arbitrary. Since

$$\lim_{n\to+\infty} \frac{p - 2^{-n}q}{m(1-2^{-n})} = \frac{p}{m} \in (0,+\infty),$$

there exists $n \in \mathbf{N}$ such that

$$y_0 := \frac{p - 2^{-n}q}{m(1-2^{-n})} \in (0,+\infty),$$

and we have $p = 2^{-n}q + m(1-2^{-n})y_0$. Let $V = W + m(1-2^n)y_0$, where $W = \{2^n u : u \in U\}$. Then V is open and $q \in V$. Hence, if $v \in V$, then $v = 2^n u_0 + m(1-2^n)y_0$ for some $u_0 \in U$, and thus $u_0 = 2^{-n}v + m(1-2^{-n})y_0$. Therefore, again by Corollary 3.2,

$$f(u_0) \leq c_m^{-n} f\left(\left[\frac{c_m}{2}\right]^n v\right) + m\left(\frac{2^n - 1}{c_m^n}\right) f\left(\left[\frac{c_m}{2}\right]^n y_0\right) + \frac{(m+2)c_m\varepsilon}{c_m - 2}.$$

By using now the sub-homogeneity of f, we obtain

$$M \leq f(u_0) \leq 2^{-n}f(v) + m(1-2^{-n})f(y_0) + \frac{(m+2)c_m\varepsilon}{c_m - 2};$$

this is,

$$f(v) \geq 2^n M + m(1-2^n)f(y_0) - \frac{2^n(m+2)c_m\varepsilon}{c_m - 2}.$$

Thus f is locally bounded from below on $(0,+\infty)$.

Proposition 3.5 *Let $f : (0,+\infty) \to \mathbf{R}$ be a sub-homogeneous function in $J_{\varepsilon,m}(+\infty)$, with $m \in (0,1)$. If f is locally bounded from above at a point $p \in (0,+\infty)$, then f is locally bounded from below at this point.*

Proof. Let $U \subset (0, +\infty)$ be an open set such that $p \in U$ and $f(u) \leq M_1$ for all $u \in U$, with $M_1 \in \mathbf{R}^+$. By Proposition 3.3, f is also locally bounded from above at the point $(c_m - 1)p \in (0, +\infty)$. Let $V \subset (0, +\infty)$ be an open set such that $(c_m - 1)p \in V$ and $f(v) \leq M_2$ for all $v \in V$, with $M_2 \in \mathbf{R}^+$. We consider $W = U \cap V'$, where $V' = \{c_m p - v : v \in V\}$. Then W is an open set and $p \in W$. Now, if $w \in W$, then $w = c_m p - v_0$ for some $v_0 \in V$, and thus,

$$f(p) = f\left(\frac{v_0 + w}{c_m}\right) \leq \frac{f(v_0) + f(w)}{c_m} + \varepsilon.$$

Therefore, $f(w) \geq c_m f(p) - f(v_0) - c_m\varepsilon \geq c_m f(p) - M_2 - c_m\varepsilon$. Thus, f is locally bounded from below at the point p.

From Propositions 3.3, 3.4 and 3.5, it follows immediately the next result.

Corollary 3.6 *Let $f : (0, +\infty) \to \mathbf{R}$ be a sub-homogeneous function in $J_{\varepsilon,m}(+\infty)$, with $m \in (0, 1)$. If f is locally bounded from above at a point $p \in (0, +\infty)$, then f is locally bounded on $(0, +\infty)$.*

Theorem 3.7 *Let $f : (0, +\infty) \to \mathbf{R}$ be a sub-homogeneous function in $J_{\varepsilon,m}(+\infty)$. If f is δ_0-m-convex, then f is δ_n-m-convex for all $n \geq 1$, where $\delta_n = \dfrac{\delta_{n-1}}{c_m} + \varepsilon$.*

Proof. The proof is by induction on n. For $n = 1$, let $x, y > 0$ be arbitrary. Thus, if $0 \leq t \leq \dfrac{1}{c_m}$, then $0 \leq c_m t \leq 1$ and we have

$$\begin{aligned}
f(tx + m(1-t)y) &= f\left(tx + \frac{(m+1)}{c_m}y - mty\right) \\
&= f\left(\frac{c_m tx + m(1 - c_m t)y + y}{c_m}\right) \\
&\leq \frac{f(c_m tx + m(1 - c_m t)y) + f(y)}{c_m} + \varepsilon \\
&\leq \frac{c_m t f(x) + m(1 - c_m t)f(y) + \delta_0 + f(y)}{c_m} + \varepsilon \\
&= tf(x) + m\left(\frac{1}{c_m} - t + \frac{1}{mc_m}\right)f(y) + \frac{\delta_0}{c_m} + \varepsilon \\
&= tf(x) + m(1-t)f(y) + \delta_1.
\end{aligned}$$

If $\dfrac{1}{c_m} < t \leq 1$, then $0 \leq c_m m(1-t) < 1$, and therefore,

$$\begin{aligned}
f(tx+&m(1-t)y)\\
&= f\left(\frac{c_m m(1-t)y + m(1-c_m m(1-t))x - m(1-c_m m(1-t))x + c_m tx}{c_m}\right)\\
&= f\left(\frac{c_m m(1-t)y + m(1-c_m m(1-t))x + (c_m t + m^2 - c_m m^2 t)x}{c_m}\right)\\
&\leq \frac{f(c_m m(1-t)y + m(1-c_m m(1-t))x) + f((c_m t + m^2 - c_m m^2 t)x)}{c_m} + \varepsilon.
\end{aligned}$$

But, by assumption, also it follows that $c_m t + m^2 - c_m m^2 t \geq 1$. Hence, by using the δ_0-m convexity and sub-homogeneity of f, we have

$$\begin{aligned}
&f(tx+m(1-t)y)\\
&\leq \frac{c_m m(1-t)f(y) + m(1-c_m m(1-t))f(x) + \delta_0 + (c_m t + m^2 - c_m m^2 t)f(x)}{c_m} + \varepsilon\\
&= m(1-t)f(y) + m\left(\frac{1}{c_m} - m(1-t) + \frac{t}{m} + \frac{m}{c_m} - mt\right)f(x) + \frac{\delta_0}{c_m} + \varepsilon\\
&= tf(x) + m(1-t)f(y) + \delta_1.
\end{aligned}$$

Therefore, the result is true for $n = 1$.

We assume now that the result is true for n. The proof for $n+1$ is similar to the proof for $n = 1$, with δ_n replaced by δ_0.

Corollary 3.8 *Let $f : (0, +\infty) \to \mathbf{R}$ be a sub-homogeneous function in $J_{\varepsilon,m}(+\infty)$. If f is δ-m-convex, then it is $c_m\varepsilon$-m-convex.*

Proof. By Theorem 3.7, for all $n \in \mathbf{N}$ we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) + \delta_n.$$

By passing to the limit as $n \to \infty$, and since $\delta_n \to \dfrac{c_m\varepsilon}{c_m - 1} \leq c_m\varepsilon$, the conclusion follows.

Proposition 3.9 *Let $f : (0, +\infty) \to \mathbf{R}$ be a sub-homogeneous function in $J_{\varepsilon,m}(+\infty)$, with $m \in (0, 1)$. If f is locally bounded from above at a point $p \in (0, +\infty)$, then f is $c_m\varepsilon$-m-convex.*

Proof. By Corollary 3.6, f is locally bounded on $(0, +\infty)$. Let $x, y \in (0, +\infty)$ be given. Without loss of generality we can assume that $x \leq my$, and we consider the interval $[x, my]$. Then there exist $M \in \mathbf{R}^+$ such that $|f(z)| \leq M$ for all $z \in [x, my]$. From this fact follows the inequalities $f(tx + m(1-t)y) \leq$

M, $-tf(x) \leq tM$ and $-(1-t)f(my) \leq (1-t)M$ for all $t \in [0,1]$, which in turn implies

$$f(tx+m(1-t)y) \leq tf(x)+(1-t)f(my)+2M \leq tf(x)+m(1-t)f(y)+(m+1)\varepsilon+2M.$$

Therefore, $f\mid_{[x,my]}$ is $((m+1)\varepsilon+2M)$-m-convex. Hence, by Corollary 3.8 (with $\delta = (m+1)\varepsilon + 2M$), f is $c_m\varepsilon$-m-convex. Because x, y are arbitrary, this finishes the proof.

It is clear that if $f : [0,+\infty) \to \mathbf{R}$ is a function ε-m-convex, then $f \in W_{\varepsilon,m}[+\infty)$. Moreover, the following result holds.

Proposition 3.10 *Let $f : (0,+\infty) \to \mathbf{R}$ be a function given and $m \in (0,1]$. If $f \in W_{\varepsilon,m}(+\infty)$, then $f \in J_{\varepsilon,m}(+\infty)$.*

Proof. By taking $t = \dfrac{1}{c_m} = \dfrac{m}{m+1}$ in (2), we have for all $x, y \in (0,+\infty)$

$$f\left(\frac{x}{c_m}+\frac{y}{c_m}\right)+f\left(\frac{x}{c_m}+\frac{y}{c_m}\right) \leq \frac{2}{c_m}[f(x)+f(y)]+2\varepsilon.$$

Hence,

$$f\left(\frac{x+y}{c_m}\right) \leq \frac{f(x)+f(y)}{c_m}+\varepsilon.$$

Theorem 3.11 *Let $f : (0,+\infty) \to \mathbf{R}$ be a sub-homogeneous function in $W_{\varepsilon,m}(+\infty)$ and $m \in (0,1)$. If f is locally bounded from below at a point $p \in (0,+\infty)$, then for $x, y > 0$ fixed but arbitrary, the function $g : [0,1] \to \mathbf{R}$, defined by $g(t) = f(tx+m(1-t)y)$, is bounded and $\dfrac{c_m\varepsilon}{2}$-midconvex.*

Proof. Since $f \in W_{\varepsilon,m}(+\infty)$, by Proposition 3.10, $f \in J_{\varepsilon,m}(+\infty)$. This fact and Proposition 3.4 in turn imply that f is locally bounded from below on $(0,+\infty)$. Hence, f is bounded from below on the closed interval $I = [\min\{x, m^2y\}, \max\{\frac{x}{m}, my\}]$ (say by $M \in \mathbf{R}^-$). Moreover, g is locally bounded from below on $[0,1]$. Thus, also by the compactness of $[0,1]$, g is bounded from below on $[0,1]$. Well then, by the sub-homogeneity (note that $m \leq t+m(1-t) \leq 1$), the ε-Wright m-convexity of f, and the fact that $\dfrac{tx+m^2(1-t)y}{t+m(1-t)} \in I$

for all $t \in [0,1]$, we have

$$\begin{aligned}
g\left(\frac{m(1-t)}{t+m(1-t)}\right) &= f\left(\frac{m(1-t)x+mty}{t+m(1-t)}\right)\\
&= f\left(\frac{m(1-t)x+t(my)}{t+m(1-t)}\right)+f\left(\frac{tx+m(1-t)(my)}{t+m(1-t)}\right)-f\left(\frac{tx+m^2(1-t)y}{t+m(1-t)}\right)\\
&\le \frac{f(tx+m(1-t)(my))+f(m(1-t)x+t(my))}{t+m(1-t)}-M\\
&\le f(x)+f(my)+\frac{2\varepsilon}{t+m(1-t)}-M\\
&\le f(x)+f(my)+\frac{2\varepsilon}{m}-M
\end{aligned}$$

Thus,

$$g\left(\frac{m(1-t)}{t+m(1-t)}\right) \le g(1)+g(0)+\frac{2\varepsilon}{m}-M.$$

Since the function $t \mapsto \dfrac{m(1-t)}{t+m(1-t)}$ is a bijection of $[0,1]$ on itself, we obtain that g is bounded from above on $[0,1]$.

Let us see now that g is $\dfrac{c_m\varepsilon}{2}$-midconvex. In fact, for all $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
g\left(\frac{t_1+t_2}{2}\right) &= f\left(\frac{t_1+t_2}{2}x+m\left(1-\frac{t_1+t_2}{2}\right)y\right)\\
&= f\left(\frac{c_m}{2}\left[\frac{(t_1+t_2)x+m(2-(t_1+t_2))y}{c_m}\right]\right)\\
&\le \frac{c_m}{2}f\left(\frac{t_1x+m(1-t_1)y+t_2x+m(1-t_2)y}{c_m}\right)\\
&\le \frac{c_m}{2}\left[\frac{f(t_1x+m(1-t_1)y)+f(t_2x+m(1-t_2)y)}{c_m}+\varepsilon\right]\\
&= \frac{g(t_1)+g(t_2)}{2}+\frac{c_m\varepsilon}{2}.
\end{aligned}$$

The following two results were proved by Ng and Nikodem in [6] (see also [2]).

Theorem. *Let X be a real vector space, and D an open and convex subset of X. If $f : D \to \mathbf{R}$ is β-midconvex and locally bounded from above at a point of D, then f is 2β-convex.*

Lemma. *Let $I \subset \mathbf{R}$ be an interval. If $f : I \to \mathbf{R}$ is β-midconvex on I and 2β-convex in the interior of I, then f is 2β-convex on I.*

From this couple of previous results, by taking $D = (0,1)$, $I = [0,1]$, $\beta = \frac{c_m\varepsilon}{2}$, and g instead of f, we have obtained an immediate corollary of Theorem 3.11.

Corollary 3.12 *The function g given in the Theorem 3.11 is $c_m\varepsilon$-convex on* $[0,1]$.

From Theorem 3.11 and Corollary 3.12, we have the next result.

Proposition 3.13 *Let $f : (0,+\infty) \to \mathbf{R}$ be a sub-homogeneous function in $W_{\varepsilon,m}(+\infty)$ and $m \in (0,1]$. If f is locally bounded from below at a point $p \in (0,+\infty)$, then f is $c_m\varepsilon$-m-convex.*

Proof. Let $x, y > 0$ and we define as before the function $g : [0,1] \to \mathbf{R}$ given by $g(t) = f(tx + m(1-t)y)$. By Corollary 3.12, g is $c_m\varepsilon$-convex on $[0,1]$. Therefore,

$$\begin{aligned} f(tx + m(1-t)y) = g(t) &= g(t \cdot 1 + (1-t) \cdot 0) \\ &\leq tg(1) + (1-t)g(0) + c_m\varepsilon \\ &= tf(x) + (1-t)f(my) + c_m\varepsilon \\ &\leq tf(x) + m(1-t)f(y) + (m+1)\varepsilon + c_m\varepsilon \\ &= tf(x) + m(1-t)f(y) + (m+1)c_m\varepsilon. \end{aligned}$$

Hence, f is $(m+1)c_m\varepsilon$-m-convex. Thus, by Corollary 3.8 (with $\delta = (m+1)c_m\varepsilon$), f is $c_m\varepsilon$-m-convex.

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