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Regularity for Solutions to Obstacle Problems of Some Anisotropic Elliptic Systems

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Abstract

This paper deals with $\mathcal{K}^{(p_i)}_{\psi,\theta}$-obstacle problems of the system of N partial differential equations

$$\sum_{i=1}^{n} D_i(a_i^\alpha(x, Du(x))) = \sum_{i=1}^{n} D_i F_i^\alpha(x), \quad \alpha = 1, \cdots N.$$

We show that, for any fixed $\beta \in \{1, \cdots, N\}$, higher integrability of the datum $\theta_*^\beta = \max\{\psi^\beta, \theta^\beta\}$ forces the component u^β of solutions u to have higher integrability as well, provided we assume suitable ellipticity and growth conditions on a_i^α.

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1 Introduction and Statement of Result

In a recent paper [1], Leonetti and Petricca considered the system of N partial differential equations

$$\sum_{i=1}^{n} D_i(a_i^\alpha(x, Du(x))) = 0, \ x \in \Omega, \ \alpha = 1, 2, \cdots, N. \tag{1.1}$$

Under the boundary condition

$$u(x) = u_*(x), \quad x \in \partial\Omega, \tag{1.2}$$

the authors showed that higher integrability of the boundary datum u_* forces solutions u to have higher integrability as well, provided a_i^α satisfy suitable ellipticity and growth conditions. Among all the results, they obtained a theorem as follows, see [1, Theorem 1.3].

Theorem 1.1 *Let* $u \in u_* + W_0^{1,(p_i)}(\Omega; R^N)$ *verify*

$$\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N a_i^\alpha(x, Du(x)) D_i \varphi^\alpha(x) dx = 0, \quad \forall \varphi \in W_0^{1,(p_i)}(\Omega; R^N) \tag{1.3}$$

under growth condition

$$|a_i^\alpha(x, z)| \le c(1 + |z_i|)^{p_i - 1}, \quad \forall \alpha \in \{1, \cdots, N\} \tag{1.4}$$

and monotonicity condition

$$\tilde{\nu} \sum_{i=1}^n |z_i^\alpha - \tilde{z}_i^\alpha|^{p_i} \le \sum_{i=1}^n (a_i^\alpha(x, z) - a_i^\alpha(x, \tilde{z}))(z_i^\alpha - \tilde{z}_i^\alpha), \quad \forall \alpha \in \{1, \cdots, N\}. \tag{1.5}$$

Then

$$u \in u_* + L^t_{weak}(\Omega; R^N),$$

where

$$t = \frac{\bar{p}\bar{p}^*}{\bar{p} - b\bar{p}^*},$$

with

$$0 < b \le \min_{i=1,\cdots,n} \frac{q_i - p_i}{q_i} \quad \text{and} \quad b < \frac{\bar{p}}{\bar{p}^*}.$$

In this paper, we consider a more general problem. We refer the reader to [1] for the notations and symbols used in this paper. Let us consider the following system of N partial differential equations

$$\sum_{i=1}^n D_i(a_i^\alpha(x, Du(x))) = \sum_{i=1}^n D_i F_i^\alpha(x), \quad \alpha = 1, \cdots, N, \tag{1.6}$$

and suppose that the Carathéodory functions $a_i^\alpha(x, z) : \Omega \times \mathrm{R}^n \to \mathrm{R}$ satisfy the growth condition (1.4) and the monotonicity condition (1.5). Let $\Psi = (\psi^1, \cdots, \psi^N)$ be any vector in Ω with values in $(\mathrm{R} \cup \{\pm\infty\})^N$ and $\theta \in W^{1,1}(\Omega; \mathrm{R}^N)$ such that

$$\theta_*^\alpha = \max\{\psi^\alpha, \theta^\alpha\} \in \theta^\alpha + W_0^{1,(q_i)}(\Omega), \quad \forall \alpha \in \{1, \cdots, N\}, \ q_i > p_i. \tag{1.7}$$

We introduce

$$\mathcal{K}_{\Psi,\theta}^{(p_i)}(\Omega; \mathrm{R}^N) = \left\{ v \in W^{1,(p_i)}(\Omega; \mathrm{R}^N) : v^\alpha \ge \psi^\alpha, \ \text{a.e.}, \alpha = 1, \cdots, N, \ \text{and } v \in \theta + W_0^{1,(p_i)}(\Omega; \mathrm{R}^N) \right\}.$$

The main theorem of this paper is the following theorem.

Theorem 1.2 *Let* $q_i \in (p_i, +\infty)$, $i = 1, 2, \cdots, n$, b *is any number verifying*

$$0 < b \le \min_{i=1,\cdots,n} \frac{q_i - p_i}{q_i}, \tag{1.8}$$

the vectors Ψ *and* θ *satisfy (1.7),* $F_i^\alpha \in L^{\frac{q_i}{p_i-1}}(\Omega)$, $i = 1, \cdots, n$, $\alpha = 1, \cdots, N$, *and* $\overline{p} < n$. *Let* $u \in \mathcal{K}_{\psi,\theta}^{(p_i)}(\Omega; R^N)$ *such that*

$$\begin{aligned} &\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N a_i^\alpha(x, Du(x))(D_i v^\alpha(x) - D_i u^\alpha(x))dx \\ \ge\ &\int_\Omega \sum_{i=1}^n \sum_{\alpha=1}^N F_i^\alpha(x)(D_i v^\alpha(x) - D_i u^\alpha(x))dx. \end{aligned} \tag{1.9}$$

For any $\beta \in \{1, \cdots, N\}$, *we have*

(i) $b < \frac{\bar p}{\bar p^*} \Rightarrow u^\beta \in \theta_*^\beta + L^t_{weak}(\Omega)$, *where* $t = \frac{\bar p \bar p^*}{\bar p - b\bar p^*}$;

if $\frac{\bar p}{(\bar p)^*} < \min_{1 \le in \le n} \left\{1 - \frac{p_i}{q_i}\right\}$, *then*

(ii) $b = \frac{\bar p}{\bar p^*} \Rightarrow e^{\eta|u^\beta - \theta_*^\beta|} \in L^1(\Omega)$ *for some* $\eta > 1$;

(iii) $\frac{\bar p}{\bar p^*} < b \le \min_{i=1,\cdots,n} \frac{q_i - p_i}{q_i} \Rightarrow u^\beta \in \theta_*^\beta + L^\infty(\Omega)$.

Note that for the special case when $F_i^\alpha(x) = 0$, $i = 1, \cdots, n$, $\alpha = 1, \cdots, N$ and $\psi^\alpha(x) = -\infty$, $\alpha = 1, \cdots, N$, the conditions in Theorem 1.2 are the same as the ones in Theorem 1.1. In this time, the result (i) of Theorem 1.2 is the same as Theorem 1.1. Thus Theorem 1.2 can be regarded as a generalization of Theorem 1.1. For some other results related to anisotropic elliptic equations and systems, we refer the reader to [2-7].

In the proof of Theorem 1.2 we will used the following lemma, see [7, Proposition 2.2].

Lemma 1.1 *Let* $v \in W_0^{1,(p_i)}(\Omega)$, *and let* $M > 0, \gamma > 0$, *and* $k_0 \ge 0$. *Let for every* $k > k_0$,

$$\int_{\{|v| \ge k\}} \sum_{i=1}^n |D_i v|^{p_i} dx \le M \left|\{|v| \ge k\}\right|^{\gamma \bar p / \bar p^*}.$$

Then the following assertions hold:

(i) *if* $\gamma < 1$, *then* $v \in L_{weak}^{\bar p^*/(1-\gamma)}(\Omega)$;

(ii) *if* $\gamma = 1$, *then there exists* $\eta > 0$ *such that* $e^{\eta|v|} \in L^1(\Omega)$;

(iii) *if* $\gamma > 1$, *then* $v \in L^\infty(\Omega)$.

2 Proof of Theorem 1.1

Let us fix $\beta \in \{1, \cdots, N\}$ and we take $v = (u^1, \cdots, u^{\beta-1}, v^\beta, u^{\beta+1}, \cdots, u^N)$, where, for $L > 0$, we have defined

$$v^\beta = \theta_*^\beta + T_L(u^\beta - \theta_*^\beta).$$

Now $v \in \mathcal{K}^{(p_i)}_{\psi,\theta}(\Omega;\mathrm{R}^N)$. Indeed, we need only to show that $v^\beta \geq \psi^\beta$ and $v^\beta \in \theta^\beta + \in W^{1,(p_i)}_0(\Omega)$. For the case $u^\beta - \theta^\beta_* < -L$, we have $T_L(u^\beta - \theta^\beta_*) = -L$, then $v^\beta = \theta^\beta_* - L > u^\beta \geq \psi^\beta$; for the case $u^\beta - \theta^\beta_* \geq -L$, we have $v^\beta = \theta^\beta_* + \min\{k, u^\beta - \theta^\beta_*\} \geq \min\{\theta^\beta_*, u^\beta\} \geq \psi^\beta$. Since $u^\beta = \theta^\beta_* = \theta^\beta$ on $\partial\Omega$, then $v^\beta = \theta^\beta_* = \theta^\beta$ on $\partial\Omega$.

It is easy to see that

$$v^\beta - u^\beta = \Big(\theta^\beta_* - u^\beta - T_L(\theta^\beta_*) - u^\beta\Big)\cdot 1_{\{|u^\beta - \theta^\beta_*|\geq L\}} \tag{2.1}$$

and

$$D_i v^\beta - D_i u^\beta = \Big(D_i\theta^\beta_* - D_i u^\beta\Big)\cdot 1_{\{|u^\beta - \theta^\beta_*|\geq L\}}, \tag{2.2}$$

where $1_{\{|u^\beta - \theta^\beta_*|\geq L\}}$ is the characteristic function for the set $\{|u^\beta - \theta^\beta_*| \geq L\}$. By inserting the test function v into (1.9) and noticing (2.1) and (2.2), we derive

$$\begin{aligned} &\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} a^\beta_i(x, Du)(D_i\theta^\beta_* - D_i u^\beta)dx \\ \geq\ &\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} F^\beta_i(D_i\theta^\beta_* - D_i u^\beta)dx. \end{aligned} \tag{2.3}$$

The coercivity condition (1.5) allows us to write

$$\begin{aligned} &\tilde{\nu}\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|>L\}} |D_i u^\beta - D_i\theta^\beta_*|^{p_i}dx \\ \leq\ &\int_{\{|u^\beta - \theta^\beta_*|\geq L\}} \sum_{i=1}^n (a^\beta_i(x, Du) - a^\beta_i(x, D\theta_*))\cdot(D_i u^\beta - D_i\theta^\beta_*)dx, \end{aligned}$$

which together with (2.3) implies

$$\begin{aligned} &\tilde{\nu}\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} |D_i u^\beta - D_i\theta^\beta_*|^{p_i}dx \\ \leq\ &-\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} a^\beta_i(x, D\theta_*)\cdot(D_i u^\beta - D_i\theta^\beta_*)dx \\ &+\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} F^\beta_i(D_i u^\beta - D_i\theta^\beta_*)dx \\ :=\ &I_1 + I_2. \end{aligned} \tag{2.4}$$

We now use anisotropic growth (1.4) and Young inequality obtaining

$$\begin{aligned} |I_1| \ &\leq\ c\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} (1 + |D_i\theta_*|)^{p_i - 1}\, |D_i u^\beta - D_i\theta^\beta_*|dx \\ &\leq\ cC(\varepsilon)\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} (1 + |D_i\theta_*|^{p_i})dx \\ &\quad +c\varepsilon\sum_{i=1}^n \int_{\{|u^\beta - \theta^\beta_*|\geq L\}} |D_i u^\beta - D_i\theta^\beta_*|^{p_i}dx, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} |I_2| &\le c\sum_{i=1}^n \int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |F_i^\beta||D_iu^\beta - D_i\theta_*^\beta|dx \\ &\le cC(\varepsilon)\sum_{i=1}^n \int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |F_i^\beta|^{p_i'}dx + c\varepsilon \sum_{i=1}^n \int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |D_iu^\beta - D_i\theta_*^\beta|^{p_i}dx. \end{aligned} \tag{2.6}$$

Combining (2.4)-(2.6) and taking ε small enough such that $2c\varepsilon < \tilde{\nu}$ we arrive at

$$\begin{aligned} &\sum_{i=1}^n \int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |D_iu^\beta - D_i\theta_*^\beta|^{p_i}dx \\ \le\ & C\sum_{i=1}^n \int_{\{|u^\beta-\theta_*^\beta|\ge L\}} ((1+|D_i\theta_*|)^{p_i} + |F_i^\beta|^{p_i'})dx, \end{aligned} \tag{2.7}$$

where C is a constant depends only on $c, \tilde{\nu}, p_1, \cdots, p_n$ and n. Let t_i be such that

$$p_i < t_i \le q_i.$$

We use the Hölder inequality as follows

$$\begin{aligned} &\int_{\{|u^\beta-\theta_*^\beta|\ge L\}} (1+|D_i\theta_*|)^{p_i}dx \\ \le\ & \left(\int_{\{|u^\beta-\theta_*^\beta|\ge L\}} (1+|D_i\theta_*|)^{t_i}dx\right)^{\frac{p_i}{t_i}} \left|\{|u^\beta-\theta_*^\beta|\ge L\}\right|^{1-\frac{p_i}{t_i}}. \end{aligned} \tag{2.8}$$

We would like that the exponent $b = 1 - \frac{p_i}{t_i}$ does not depend on i. To this aim, we need only to choose $t_i = \frac{p_i}{1-b}$, $i = 1, 2, \cdots, n$. Since we need $p_i < t_i \le q_i$, we require that b satisfies (1.8). Thus (2.8) becomes

$$\int_{\{|u^\beta-\theta_*^\beta|\ge L\}} (1+|D_i\theta_*|)^{p_i}dx \le A_i \left|\{|u^\beta-\theta_*^\beta|\ge L\}\right|^{b}, \tag{2.9}$$

where

$$A_i = \left(\int_{\{|u^\beta-\theta_*^\beta|\ge L\}} (1+|D_i\theta_*|)^{t_i}dx\right)^{\frac{p_i}{t_i}}.$$

Similarly,

$$\begin{aligned} \int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |F_i^\beta|^{p_i'}dx &\le \left(\int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |F_i^\beta|^{\frac{t_i}{p_i-1}}dx\right)^{\frac{p_i}{t_i}} \left|\{|u^\beta-\theta_*^\beta|\ge L\}\right|^{1-\frac{p_i}{t_i}} \\ &= B_i \left|\{|u^\beta-\theta_*^\beta|\ge L\}\right|^{b}, \end{aligned} \tag{2.10}$$

where

$$B_i = \left(\int_{\{|u^\beta-\theta_*^\beta|\ge L\}} |F_i^\beta|^{\frac{t_i}{p_i-1}}dx\right)^{\frac{p_i}{t_i}}.$$

Substituting (2.9) and (2.10) into (2.8) we arrive at

$$\sum_{i=1}^{n}\int_{\{|u^{\beta}-\theta_{*}^{\beta}|\geq L\}}|D_i u^{\beta}-D_i\theta_{*}^{\beta}|^{p_i}dx\leq C\sum_{i=1}^{n}(A_i+B_i)\left|\{|u^{\beta}-\theta_{*}^{\beta}|\geq L\}\right|^{b}. \quad (2.10)$$

Theorem 1.2 follows from Lemma 1.1.

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References

[1] F.Leonetti, P.V.Petricca, Global summability for solutions to some anisotropic elliptic systems, Adv Diff. Eqns., 2015, 11-12, 1165-1186.

[2] H.Y.Gao, Q.H.Di, D.N.Ma, Integrability for solutions to some anisotropic obstacle problems, Manuscripta Math., 2014, 146(3-4), 433-444.

[3] H.Y.Gao, C.Liu, H.Tian, Remarks on a paper by Leonetti and Siepe, J. Math. Anal. Appl., 2013, 401(2), 881-887.

[4] H.Y.Gao, Q.H.Huang, Local regularity for solutions of anisotropic obstacle problems, Nonlinear Analysis, 2012, 75(13), 4761-4765.

[5] H.Y.Gao, Y.M.Chu, Quasiregular mappings and $\mathcal{A}$-harmonic equations, Science Press, Beijing, 2013. (In Chinese)

[6] F.Leonetti, F.Siepe, Integrability for solutions to some anisotropic elliptic equations, Nonlinear Analysis, 2012, 75, 2867-2873.

[7] A.A.Kovalevsky, Integrability and boundedness of solutions to some anisotropic problems, J. Math. Anal. Appl., 2015, 432, 820-843.

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